A Handbell Compendium

a summary *by* Michael Jedamzik

Handbell Compendium

a summary

Michael Jedamzik

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Michael Jedamzik has been ringing handbells since 2001 and participates in the Handglockenchor Wiedensahl (German, "Handbell Choir Wiedensahl"), see [Eic10, *retrieved* December 31, 2010]. Michael Jedamzik was born in Stadthagen, Germany. In 2011 he graduated with a Bachelor of Science degree (Mathematics) at the Gottfried Wilhelm Leibniz Universität Hannover (former names: Königliche Technische Hochschule, Technische Hochschule Hannover, Technische Universität Hannover, Universität Hannover), Germany.

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THIS book is dedicated to director Thomas Eickhoff, co-director Heinrich-Arend Krömer d. J., and to the *Handglockenchor Wiedensahl* for excusing my flaws when playing handbells and allowing me to be part of the ensemble.

Pagina vacat.

We hold these truths to be self-evident, that all ringers are created equal, that they are engaged by their directors with certain unalienable duties, that among these are rehearsals, knowledge in ringing and the pursuit for music.

from Declaration of a director
THOMAS JEFFBELLSON

Foreword

The most delightful and fulfilling aspect of the handbell world is its people. The music lifts us up, events help us to grow and learn, the sound of the bells is inherently magical and touches our hearts. But it is the people of the handbell world – the composers, conductors, organizers, teachers, and above all the ringers – that motivate and inspire us and become blessings in our lives.

Handbell people are enthusiasts, loyal devotees to this wonderful art form and to the people they ring with and for. They cheer one another on and become a family. No other music so completely depends on a close-knit, energetic team. Handbell people encourage each other with a generous spirit, a quirky sense of humor, and genuine love for becoming better than the sum of their parts through making music together.

Although Michael Jedamzik doesn't make handbells part of his professional career, he is a ringer who LOVES handbells! I have been delighted to meet Michael through the global handbell network and through reading his "Handbell Compendium", I have been educated by his extensive and earnest research. He has compiled information from a huge number of sources that will interest scientists, musicians in endeavors other than handbells, handbell leaders, ringers, and listeners. I have been impressed at his creation of a reference work that belongs in the library of everyone who leads or just plain loves handbells.

But most of all, I have been inspired by a ringer who is one of the handbell world's truly great enthusiasts. Michael LOVES handbells. And he has created an enduring and valuable work for all of us who share that love.

Kevin McChesney
Professional Handbell Composer/Arranger and Conductor
Colorado Springs, September 18, 2016

Kevin McChesney graduated with highest honors from the *University of Colorado* at Boulder with a BMus in Composition and Theory. A composer and arranger of handbell music, Kevin currently has over 900 titles in print and is one of the very few musicians who makes handbells a full-time vocation. He has won numerous awards for his work, including winning *American Guild of English Handbell Ringers Composition Contests* and *Jeffers Composer of the Year*. Kevin is the handbell editor for *Jeffers Handbell Supply* and the *RingingWord catalog*. He is also co-founder of the *Solo To Ensemble Project*, STEP, http://www.sonologymusic.com/. He is Music Director of one of the premier handbell concert groups the *Pikes Peak Ringers* of Colorado Springs and also the Artistic Advisor of the *Atlanta Concert Ringers*. His work with Pikes Peak Ringers includes the premiere and 13 further performances of his *Concerto for Handbell Choir and Orchestra* and a recording with world-renowned cellist Yo-Yo Ma. Kevin is also clinician for the *national advanced youth ringer event*, the *National Honors Handbell Ensemble*. He is in demand throughout the handbell world as a workshop clinician and festival conductor. He also shares his experiences and ideas with poems, see https://sonologymusic.com/products/limericks-lessons-and-life-in-handbells/ for his book.

Kevin lives in Colorado Springs, CO, with his wife Tracy and their cats, Pearl and Aileen.

Luckily, the table of contents is easy to write, once the book itself is written. Any more efforts are a waste of time, because most people do not read the table of contents.

from On transferring Information
Gillian Doe

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Pagina vacat.

Warum sollte ich ein Buch schreiben? Bücher gibt es doch schon für wenige Euro im Supermarkt.

from Latin Courses
Markus Kröger

0. Preface

An advanced handbell ringer needs to know many different techniques of *ringing a handbell*. Composers have been quite imaginative by inventing or introducing new techniques within their compositions. I hope to discuss the most common ones within this document and to add some basic knowledge about handbells necessary to prepare or execute a concert for handbell ringers. I must admit that some aspects might be too specific for some ringers and are included because I am interested in them. Anyway, these specific ones were used when explaining handbells to auditors with profound knowledge of physics. The aspects for preparing a concert from a director's, manager's or conductor's point of view are not covered by this book. However, learning to play handbells by instruction books seems to be quite difficult, and I advise one to learn from experienced ringers or directors, hence learning to play handbells is not covered in this document. If you try to learn handbell ringing only by literature or video, you may not notice all of your faults which can otherwise be corrected by experienced ringers. One might also consult professionals or attend a workshop given by professionals. The aspect of healthy ringing, which gained more attention recently, is not covered either.

Various references to books, web pages, and documents are provided. You can easily distinguish all sources by a parenthetical referencing, for example "[Jed17, page 16]": You just have to find the identifier [Jed17] on appendix F to get all relevant bibliographic information of the publication. In this case, the reference also offers a particular information on the page of the book. Although I have elaborated painstakingly, no responsibility is taken for the correctness of this document. I would like to get suggestions and have mistakes pointed out.

I do not intend to show preference for a brand in this book, although I enjoy playing handbells manufactured by *Schulmerich Bells* and I have a little experience with ones made by *Malmark Bellcraftsmen* and by *Whitechapel Bell Foundry*. I have not seen a handbell from *John Taylor & Co. Bellfounders, Bellhangers and Carillon Builders*, *Prima Gakki* or those manufactured by companies who no longer manufacture handbells. Founding a new ensemble or buying a full new handbell set of a different brand instead of extending the present handbell set should be seldom. Most ensembles, and in particular ringers, have no real option in changing brands. Preferring one brand seems to be a question of taste, so there is no need for quarreling.

I did choose English instead of German because this allows me to use some original quoted content and I may reach some more readers. This includes ringers and directors from the United States I know personally. Although the language of handbell ringers is naturally *music*, it is hard to transfer information as given in this book with music solely, and most of the ringers are able to read and understand simple English. Each chapter of this book begins with a (more or less) famous epigraph. However, some of them are slightly altered by me. I do not want to translate any of the non-English ones into English.

The first edition of this compendium was mostly known to my fellow ringers. While the first edition of this compendium had 56 pages, the second edition is a huge step in many ways: additional topics, several new texts, improvements and error-correction, several new tables and figures. Almost all of the figures of the second edition are new or refurbished. Some of the new figures can be seen as canonical figures filling gaps of the first edition. The second edition also features an index. The selection of new topics includes history, tone of a bell, a short discussion on the Doppler effect, the bell formula and on the geometry of a bell, handbell ringer's etiquette, and a selection of handbell related poetry as an appendix. Chapter 2 deals with elementary mathematics of physical models. All trivial calculations are omitted. Since all results are interpreted in terms of the musical world, this chapter is also interesting for musicians with mathematical anxiety.

According to [Sha14], any bell which is designed to ring in hands and has an interior-mounted clapper is named *handbell*. Since this document is dedicated to the music of a handbell type which is known in the United States of America as *English handbell*, I use the term *handbell* synonymously.

I would like to thank all contributors to TEX, since this document is realized with LuaTEX, LATEX 2$_\varepsilon$, AMS-LATEX, BIBTEX, MusiXTEX, PGF/TikZ, and the TEX Live (Version 2016-64) distribution, see [TUG93]. To represent these people I would like to thank DONALD ERVIN KNUTH, RICHARD KOCH, MARKUS KOHM, LESLIE LAMPORT, and TILL TANTAU. I further express my sincere thanks and gratitude to Dr. RUDOLF GÜNZEL for setting up the base of the Handglockenchor Wiedensahl by donating three octaves of Schulmerich handbells, to ILSE KRÖMER for introducing me to instrumental music as well as the music of handbells, to MARILYN UZZLE for correcting linguistic errors and thus being my greatest help on this book, to the wonderful C. MILTON RODGERS III. for technically assisting MARILYN, for his love of music, for teaching the first handbell ringers of the Handglockenchor Wiedensahl, for his friendship, for his hospitality, and for setting up the base of the real bass range by – as the director of the *Parrish Bells* – donating a G_2 as the first bell of the lower sixth octave of handbells, to KEVIN for his thoughtful foreword, and finally to THOMAS for every shiny and marvelous moment within his direction. THOMAS surely inspired my enthusiasm for writing this book.

Michael Jedamzik

Pagina vacat.

Fest gemauert in der Erden
Steht die Form aus Lehm gebrannt.
Heute muss die Glocke werden!
Frisch, Gesellen, seid zur Hand!
Von der Stirne heiß
Rinnen muß der Schweiß,
Soll das Werk den Meister loben!
Doch der Segen kommt von oben.

from Das Lied von der Glocke
FRIEDRICH VON SCHILLER

1. About handbells

Linguistic compounds are quite uncommon in the English language. In early publications handbells may be written as "hand-bell", see [Joh87]. This term evolved to "handbell" which is used today. When searching for older literature, different terms may be used.

To represent the corresponding handbell ensemble, every member has to know some basics about handbells particularly, with regard to being asked by auditors after a concert.

Some ringers may find the principle of handbell ensembles as follows fine for explaining to persons not familiar with handbells.

1.1. Principle of handbell music

A usual piano with 7.25 octaves (A_0 - C_8, 88 keys) is played by one pianist. The general idea of ringing handbells in an ensemble is to represent each tone with a bell rather than with a string or a group of strings, respectively.[1]

[1] It should be noted that small handbell ensembles have only two or three octaves and 7.5 octaves of handbells is usually a maximum, compare section 1.9.

Each member of a handbell ensemble, called a *ringer*, is assigned to two up to five bells. This just means that a handbell ensemble consisting of 10 to 18 ringers and one director, viewed as just one instrument[2], is more flexible than a piano played by one pianist since one pianist is more limited in playing many tones at once or playing in a sequence and using different techniques at once.

On the other hand, making music with more musicians has its challenges, too: Every note is played by exactly one musician and errors cannot corrected by others. A common problem is *vertical alignment*: All ringers have to ring at exactly the same time when ringing a chord and with proper force. A bit more subtle is *horizontal alignment*: The ringers have to keep dynamics up. Larger runs and so called *solo passages* where only one bell sounds at most can be seen as aspects of horizontal alignments and are fragile. Even if only one ringer does not play perfectly, errors are clearly noticeable. The notable difference between two tones on a piano is the position of the keys, the tone itself, and sound decay. Handbells do not only vary notably in weight, but also in how to play them. The time which the clapper takes to swing against the bell's casting and the force which is needed to ring a bell appropriately varies from bell to bell. These aspects also define the kind of rhythm and tempo combinations which are considered unplayable.

The number of ringers and assigned bells in ensembles depends upon available players and on individual pieces.

When ringers are playing handbells, melodies are literally going from one hand into another so coordination is essential. Hence, another unique problem becomes clear: a musician as a member of an orchestra can usually rehearse solely. Handbell ringing can only be rehearsed well when done with the entire handbell choir.

[2]This point of view is problematic and used to make the principle of handbell music clear: There is a debate whether a single handbell is an instrument or a full set of handbells is one instrument. A single handbell ringer as a member of a handbell ensemble can be seen as a musician similar to a percussionist of an orchestra.

1.2. Anatomy of a handbell

Each handbell consists of five components realizing that most components consist of several parts, see also figures 1 and 2:

Casting. The sounding metal part of a handbell is cast from bronze (most common), aluminium, or sometimes brass or a mix of these. To achieve a proper tuning, the casting is carefully ground and usually polished. Since the thickness is one important parameter to the tuning, it is difficult to tune the bell a second time. Since a handbell ensemble is practically unable to tune a handbell, other instruments have to be adapted to the handbells when played together. A usual reference is the handbell A_4, compare section 1.9.

Clapper. The clapper causes the oscillations in the casting, which generate the sound of the handbell. The clapper in a handbell is mounted in the handbell's yoke so that it can move in only one dimension: back and forth. Clappers in American handbells are usually adjustable by rotating the clapper head to choose a timbre. This is called *voicing*. The casting-touching boundary of the clapper head is usually made of non-metal materials such as rubber or felt with the exception of clapper heads of the upper sixth or seventh octave handbells which are often made of metal.

Yoke. The mechanical part which is located inside the handbell and the bearing is called the *yoke*.[3] The restraining spring limits the clapper's motion and usually transports the clapper to back position after a ring. A proper setting of the spring prevents the bell from *back-ringing* or *double-ringing*, so that the clapper does not touch the opposite side of the casting after ringing the bell.

Hand guard. The hand guard is a stable circular plate between the casting and the handle of a handbell. The handbell rests in the ringer's hand

[3] It is important that the needed striking force is well adjusted: Too little necessary force results in accidental rings or back rings while too much necessary force forces the ringer to use more power causing unnecessary stress to the handle which may result in a shorter life expectancy.

(in particular the thumb and forefinger) on the hand guard, and for that reason the hand guard is also called a *hand rest*.

Handle. The handbell is held at the handle by the ringer. The handle indicates the pitch of a handbell so that the bell can be identified easily while the bell lies on a table. The handle also features the *clapper indicator*, which is usually a symbol of a bell, to distinguish the front side from the back side. The symbol of the clapper indicator often is *campaniform*[4].

The hand guard and handle are features which are obviously not present on mounted bells like church bells. For one example, compare figure 3.

1.3. English and American handbells

In this chapter *American handbells* and *English handbells* are distinguished. This classification is based on the Handbell Ringers of Great Britain in [LFW07, page 3] that points out the differences. While English handbells are made of traditional materials like leather or felt, American manufacturers, by contrast, use rubber and plastic[5] for non-metal parts. Also, English handbells are tuned solely by ear in contrast to American ones. Modern materials, however, may allow construction of more elaborate handbells.

1.4. Brief history of handbells

This whole section is based on a translation of a variation of [Eic10, *Historisches*, *retrieved* June 2, 2013]. Other references are included and cited as usual. The article [But10, *retrieved* June 2, 2013] gives a more intense view of the early history of handbells in the United States of America and lists some references. A variety of history-related articles may be found in the *Overtones* by the Handbell Musicians of America. Another standard reference is [TS14].[6]

[4] Meaning: in shape of a bell.
[5] Schulmerich replaced the plastic handles of the lower seventh octave with metal handles before 2014.
[6] Like other areas of study, the history of handbells is a controversy.

1.4.1. Early uses of bells

The handbell as an instrument was already known in China about 1 600 b. C.. Buddhism helped to spread the bell as an instrument to Asia Minor. Crusaders brought the bells from Orient to Occident. The first written evidence of bells used in music dates back to the 13^{th} century. Bells were probably used in processions in the Dark Ages and in the Renaissance due to their brilliant sound.

1.4.2. References in the Bible

There are numerous references of bells or cymbals in the Bible. For *bell*[7] see EXODUS 28:33-35. 39:25-26, 2 SAMUEL 6:5, ZECHARIAH 14:20, (apocryphal) SIRACH 45:9-11, and 1 CORINTHIANS 13:1. For *cymbal*, see 1 SAMUEL 18:6, 1 CHRONICLES 13:8. 15:16-19,28. 16:5,42. 25:1-6, 2 CHRONICLES 5:12-13. 29:25, EZRA 3:10, NEHEMIAH 12:27, PSALM 150:5, (apocryphal) JUDITH 16:1-2, and (apocryphal) 1 MACCABEES 4:54. 13:51, see also appendix C.3.3.

Note: Many translations differ slightly in verse counting, especially on the apocryphal texts. Some online bible websites offer search tools.

1.4.3. Handbells in Great Britain

The first handbells in today's style were invented by *change ringers* in Great Britain as a practicing tool. The basic idea of *change ringing* is ringing several elements of the symmetric group $\mathfrak{S}_n$ for small $n \in \mathbb{N}_{>1}$, that are permutations of n numbers, where each bell represents a number, see figure 4 for one example of $\mathfrak{S}_4$. Table 1 shows the number of permutations. These bells usually were set up in church towers and were rung via ropes. The handbells helped in practicing by switching from rather large immobile and tower-mounted bells to small handbells which made it more comfortable on cold days since the ringers could practice almost anywhere and not only in the cold church tower. The neighbors were not bothered by extensive practicing either. The basic design of a clapper, which can only swing back and forward, was used in the first handbells and is based on the clapper

[7] Dr. MARTIN LUTHER chose the word *Schelle*, not *Glocke* while the Elberfelder translation mostly uses the diminutive *Glöckchen*.

mechanism of the bells in change ringing, see [Sha14]. See [BBA94] for some more information on change ringing.

Handbells were improved. To improve the sound, wooden-made and finally leather-made handles were introduced. Innovative yoke and clapper were introduced to control the strike. Change ringers soon discovered another kind of music which could be performed with the now born *handbells*: They demanded chromatic sets to ring melodies instead of merely practicing change ringing. The Whitechapel Bell Foundry introduced such sets in the late 17th century.[8] The handbell movement reached its climax in the mid 18th century. Massed ringing with up to two-hundred bells had already been introduced. The repertoire already consisted of opera tunes, chorales and popular music. Annual competitions motivated many ensembles to participate. The First World War caused this era to decline and finally to end. Manchester hosted the last British competition in 1925.

The Scottish *St James Ringers* (Paisley, GB-RFW) are known for being founded in 1884 and being one of the oldest ensembles and still using their first set of handbells, compare [Sch04, *Handbells in Scotland*, *retrieved* June 8, 2013].

1.4.4. Handbells in the United States

The US-businessman and circus manager Phineas Taylor Barnum met the British *Lancashire Ringers*, one of the best handbell ensembles at this time in the 1840s. Barnum organized a concert tour in the eastern United States. To give them an audience-attracting touch, he named them *Swiss Bell Ringers* and they were instructed to wear Swiss costumes. They soon gained popularity. In 1895 Arthur Nichols formed the first handbell ensemble in the United States of America.

[8] From [SW01, *Handbell*, *retrieved* December 31, 2010]: "The first tuned handbells were developed by brothers Robert and William Cor in Aldbourne, Wiltshire, England, between 1696 and 1724. The Cor brothers originally made latten bells for hame boxes, but for reasons unknown, they began tuning their bells more finely to have an accurate fundamental tone, and fitted them with hinged clappers that moved only in one plane." The article [Mar97] is quoted.

His daughter MARGARET SHURCLIFF[9] made handbells even more popular in New England in the early 20th century. She initiated the creation of a dozen new handbell ensembles. The Christmas concerts of the *Beacon Hill Ringers* were the most famous handbell concerts in the early era of US-handbells. Margaret Shurcliff founded the *New England Guild of English Handbell Ringers*, the first handbell association in 1937. The continual growth of including handbells in worship services in the 1940s helped to make them even more popular. The *American Guild of English Handbell Ringers*[10] was founded in 1954, compare section 1.12 and [HMA14, *Inside the guild. A proud Tradition*, *retrieved* June 6, 2013]. According to [HMA14, *Inside the guild. A proud Tradition*, *retrieved* June 6, 2013], MARGARET SHURCLIFF said in 1955:

> "WHILE handbell ringing in England is on the wane, we are just beginning to grasp the possibilities here."

1.4.5. Birth of American handbells

Many American ensembles mostly imported handbells from Great Britain, especially from Whitechapel. In 1962 Schulmerich began manufacturing handbells. In 1974 the manufacturer Malmark was formed to produce handbells. Both are located in the Commonwealth of Pennsylvania. Since the handbells from both differ from the English ones, the *American handbell* was born, see section 1.3. The many struggles between these two manufacturers were called *the great handbell war*, see [Kes13a, Kes13b].

These two manufacturers are the global market leaders for handbells today. Other manufacturers in the United States, however, made (English) handbells before Schulmerich, see [Kea00, *Bells and Manufacturers. Are their other handbell manufactures?*, *retrieved* March 10, 2005].

[9] From [SW01, *Handbell*, *retrieved* December 31, 2010] and partially conflicting: "Handbells were first brought to the United States from England by MARGARET SHURCLIFF in 1902. She was presented with a set of 10 handbells in London by ARTHUR HUGHES, the general manager of the Whitechapel Bell Foundry after completing two separate two-and-a-half-hour change ringing peals in one day."

[10] Now *Handbell Musicians of America.*

1.4.6. Recent development

Composers have been quite imaginative with inventing or introducing new techniques within their compositions. A council consisting of members from the then AGEHR, composers and publishers developed a notation convention for handbell music.[11]

Handbells are also used in music education[12] and music therapy. Some University-students, as well as college-students participate in handbell ensembles at their university or collage. Handbells still have a place in many worship services in the United States of America.

Handbell music gained some popularity in Great Britain again, which is indicated by the formation of the society *Handbell Ringers of Great Britain* in 1976. It seems that handbells are most popular in the United States of America, East Asia and the Australian continent but are known more or less worldwide.

1.4.7. Situation in Germany

Although this section may not be interesting for ringers outside Germany, it is justified to help inform German ringers (including my fellow ringers). Compare [Eic16b] for additional information.

Handbell music was brought to Germany by US-soldiers or better liberators from the US[13] in the American Zone of Occupation after the end of World War II. The first German Handbell ensembles were formed in *Schwalmstadt* and *Aschaffenburg* in 1973 and 1979, respectively. Due to the help of US-church musicians working in Germany, some more ensembles were formed in the 1980s. Donations by US-parishes even helped founding three ensembles in the communist *German Democratic Republic*.[14]

[11] Chapter 5 deals with theses conventions. This was actually a core element of first edition of the *Handbell Compendium*.

[12] Thomas Eickhoff writes about himself using handchimes in primary education in [Eic05]. The title of the article can be translated as "Applications of handchimes in elementary school". He also hosted the workshop [Eic16a] in which he illustrated his much more improved concept and his learning achievements.

[13] See also [vW85, *retrieved* June 6, 2013] and [Knu14, *Infrequently Asked Questions*, *retrieved* November 5, 2014] for a translation.

[14] To stop confusion: After the end of the World War II, the world and Germany was split into East and West. The west German part of the iron curtain was formed into the

The effort of C. Milton Rodgers III. and Nancy Poland led to the first German handbell symposium with nine German handbell ensembles in *Eschweiler* in 1989. A meeting of directors from both German sides of the Iron Curtain led to a second symposium in 1992 organized by Jürgen Tabel and the *Ländlichen Akademie Krummhörn e. V* with 21 participating ensembles and over 250 ringers, including ringers from East Germany, Switzerland and a handbell choir of the *American Church in Paris* under the direction of Fred Gramann.

Some German handbell ensembles go on concert tours, the favorite destination are most likely the United States of America. Despite efforts by German handbell ensembles, handbells are quite unknown to the German public.

The webpages [Die16a, Die16b] lists German and European handbell ensembles.

1.4.8. ... and in Wiedensahl

The oldest known mention of the name *Wiedensahl* for a settlement is a deed from August 1, 1253 which assigns the settlement to the nearby Loccum Abbey which was founded on March 21, 1163, still exists, and nowadays is a Lutheran seminary and academy as well as a church, compare [Wie03, pages 4, 5, 7, 9] and [Hir14, page 14] and see [Loc16] for more information. Wiedensahl has approximately 1000 inhabitants.

The *St. Nicolai church* was built not later than 1275, compare [Wie98, page 28] and [Wie75, page 67]. There has been some modifications since then. See [Pee97, pages 32-40] for more information, another treatise is [Ron13]. Its two church bells were cast in 1521 and 1693, respectively, compare [Wie75, page 68]. Pastor Heinrich Brandes announced the conversion from the Roman Catholic Church to Lutheranism between 1520 and 1527, compare [Wie98, page 46]. See figure 5 for an image of the church. Note that Dr. Martin Luther published his *Dispvtatio pro declaratione virtvtis indvlgentiarvm* better known as *Ninety-five theses* in 1517.

Bundesrepublik Deutschland (Federal Republic of Germany) and the East part was formed into the *Deutsche Demokratische Republik* (German Democratic Republic). West Germany and East Germany reunited in 1990.

For the remaining part of this section, we refer to [Eic10, *Über Uns, retrieved* December 26, 2016], [Eic13, pages 3, 4], [Bor17] and [Wie03, pages 313, 314].

In 1987 Dr. Rudolf Günzel (*October 18, 1913; †September 14, 1999) donated three octaves of Schulmerich handbells to the church community. He was introduced to handbell music by a radio program on the *Revier-Glocken-Chor Bottrop* (now *RevierGlockenChor Bottrop*) and wanted to introduce this music to Wiedensahl after another institution has refused the gift and Heinrich-Arend Krömer d. Ä. has shown an interest on behalf of the St. Nicolai church community. Heinrich-Arend Krömer d. J. was the first director of the *Handglockenchor Wiedensahl*. C. Milton Rodgers III. taught the first ringers to ring handbells in 1988 and 1989. In 1995 Thomas Eickhoff became the director of the ensemble.

The *Handglockenchor Wiedensahl* participated on the symposia listed in section 1.4.7 and organized a small festival in 2002. The ensembles made concert tours to the United States of America (2003 and 2013), South Africa (2006), and East Asia (Hong Kong and Taiwan, 2015) and performed in the European Parliament, Brussels, in 2015. In 2012 and 2016 the Handglockenchor Wiedensahl became a laureate of the *Deutscher Orchesterwettbewerb* (German, “German Orchestra Competition”) by *Deutscher Musikrat* (German, “German Music Council”) two times in a row. The German public radio and television broadcaster *Norddeutscher Rundfunk* (NDR) and the German commercial television network *Radio Télévision Luxembourg Nord* (RTL Nord) broadcasted features about the ensemble on radio and television.

The ensemble currently performs on 5.5 octaves of Schulmerich handbells (F_2, G_2 - C_8) and five octaves of Schulmerich handchimes.

1.5. About handchimes

A *handchime* is a newer instrument which can be briefly described as a huge tuning fork with an external clapper, see figure 6.

> “The handchime is a metal tube, most commonly an aluminum extrusion, slotted and cut to produce a musical tone. The length of the slot in the tubing determines the fundamental pitch which is the clearly dominant tonal element, while the length of

> the unslotted portion (that which is held in the hand) is selected and fitted with a plug to provide substantial reinforcement of the tonal frequency produced by the slotted portion (the tines). The clapper mechanism is externally mounted and strikes the tube at the predetermined point to produce the desired pitch," [Kea00, *Frequently Asked Questions*, *retrieved* June 4, 2004].

Large handchimes likely do not have a clapper and are rung with a mallet. High pitched handchimes are not made of tubes and therefore do not have an air column plug, only tuned via the tuning slot and the length of the tines.

The name of handchimes differs from manufacturer to manufacturer: Schulmerich calls them *MelodyChime®*, Malmark calls them *Choirchimes®* and Suzuki uses the term *ToneChime*, see also section 1.7. Schulmerich also manufactured chimes named *MelodyWave*™. These instruments are MIDI-controllers, that is they are wireless electronic remote controls rung like handchimes to reproduce electronic sounds with a computer. A handbell ensemble thus can, for example, simulate a brass ensemble by ringing.

Handchimes are often considered as a training tool for beginners such as in elementary schools. They are less heavy, not as vulnerable to moisture as handbells and inexpensive compared to handbells. Due to their interesting sound, they are sometimes used in handbell pieces as well. Another application is using bass handchimes to support the bass line when a strong tone is required. Many techniques performed on handbells cannot be performed on handchimes, compare section 1.11.4.

1.6. Brief history of handchimes

[McG96, *Handbell*, *retrieved* June 7, 2003] gives a more detailed view on the history of handchimes. Malmark introduced *Choirchime®* in 1982. Schulmerich has manufactured handchimes called *MelodyChime®* since 1998, [Sch10, *About us. Our History*, *retrieved* April 13, 2013]. For more information see section 1.7.1.

1.7. Manufacturers

Handbells are currently manufactured on the three continents of North America, Europe and recently Asia. As mentioned in section 1.3 there are differences from company to company. Table 2 gives an overview of all current manufacturers. There is also a list of other manufacturers in [Kea00, *Bells and Manufacturers*, *retrieved* March 10, 2005]. The manufacturers complied with the request for larger bell sets and even manufactured some custom bells, example provided Basso Profundo handbells for *Westminster Concert Bell Choir* or even special bass bells by Whitechapel suitable for ringing with Schulmerich handbells for the *Southminster Ringers*, Pittsburgh, since Schulmerich did not offer these bells at that time, see [PJ13].

A visit to [Nel00, *retrieved* April 26, 2013] is recommended. The site is a project by Susan T. Nelson and they even have some interesting sound samples, for example first generation *Silver Melody Bells*™ made by Schulmerich.[15]

One Note: With respect to section 3.5.3 different fonts are used for different bell sets.

1.7.1. American handbells

The most famous manufacturers of American handbells may be *Schulmerich Bells*, see [Sch10, *retrieved* May 21, 2013], and *Malmark Bellcraftsmen*, see [Mal13a, *retrieved* May 21, 2013]. Both are located in the Commonwealth of Pennsylvania, USA. All handbells made by Schulmerich are cast by *Bridesburg Foundry* and made of 79.999 percent copper and 19.999 percent tin, hence at least 99.998 purity, see [Fri88].[16] Bronze handbells by Malmark are cast by *Bridesburg Foundry*, too, see [JW06, *Season 7, Show 29. August 5, 2012, 12:25*, *retrieved* April, 26, 2013].

Schulmerich's handchimes are called *MelodyChime*®. Schulmerich reintroduced a two octave ($\mathcal{C}_5$ - $\mathcal{C}_7$) set of Silver Melody Bells in summer 2013,

[15] Jonathan Goldstein, former President of *Schulmerich Bells*, said that there existed only 30 bell sets of these rare Bells in 2012, see [JW06, *Season 7, Show 56. October 23, 2012, 19:14*, *retrieved* April 26, 2013].

[16] A representative of Schulmerich told me on a factory tour on March 25, 2013 that all bells are still cast by Bridesburg.

see [Sch10, *Handbells. Silver Melody Bells*, *retrieved* July 31, 2013]. Silver Melody bells are cylinder-shaped, nickel-plated brass bells “providing a distinctive, bright, and robust bell timbre.,” see [Sch10, *Handbells. Silver Melody Bells*, *retrieved* July 31, 2013]. They were initially made in limited numbers to celebrate the 25^{th} anniversary of the AGEHR.

Malmark’s handchimes are called *Choirchime*® and their aluminium handbells are called *Basso Profundo*. Compared to their Malmark bronze equivalents, the aluminium handbells are lighter, larger in diameter and have a more dominant fundamental tone.

Jacob H. Malta developed all (at least the base octaves) of Schulmerich’s and Malmark’s handbells, see [Kea00, *Who’s Who in Handbells. Letter M*, *retrieved* October 14, 2004].

The *Westminster Concert Bell Choir* owns a purposely built lower ninth octave set bass handbells made of aluminium by Malmark. This ensemble is maybe the only one possessing and ringing such handbells, so they perform on 8 octaves (7 octave set and lower eighth and ninth octaves, see section 1.9), see [Mal13a, *News. Westminster Concert Bell Choir, conducted by Kathleen Ebling Shaw*, *retrieved* June 3, 2013]. The ensemble kindly allowed M. Jedamzik to ring their C_1 after one of their great concerts on April 4, 2003. The Capital City Ringers claim that Malmark has manufactured bronze handbells G_0 and A_0 which they call the largest playable handbells in the world, [Cap08, *About us*, *retrieved* December 23, 2012]. The existence was witnessed by many visitors of the factory like Douglas Benton on June 14, 2012. On April 24, 2013 he added “It is at the factory. Jake [H. Malta — M. Jedamzik] told me had $50,000 invested in this one casting, and that it would never leave the factory!”, see [Mal13b, *Photos. Posted on June 15, 2012*, *retrieved* April 29, 2013]. According to section 2.2, the fundamental frequency of those bells are $27.5 \cdot \mathsf{Hz}$ and $55 \cdot 2^{-7/6} \cdot \mathsf{Hz} \approx 24.4997 \cdot \mathsf{Hz}$, respectively. Larry J. Sue published a video of him ringing this bell on [Sue15, *retrieved* June 15, 2015]. According to [Mal13b, *Photos. Posted on April, 24 2013*, *retrieved* June 15, 2015], the weight of the A_0 is $(16 + 1/4) \cdot \mathsf{lb_m} \approx 7.371 \cdot \mathsf{kg}$, the overall height $25 \cdot \mathsf{inch} = 63.5 \cdot \mathsf{cm}$ and diameter on mouth $19 \cdot \mathsf{inch} = 48.26 \cdot \mathsf{cm}$.

The Japanese company *Prima Gakki Co. Ltd.*, see [Pri13, *retrieved* May 21, 2013], began manufacturing handbells recently[17]. Although aiming for the Japanese or Asian market, these handbells are clearly American handbells according to the definition given in section 1.3. The bells are likely pitched to $442 \cdot \mathsf{Hz}$ or $884 \cdot \mathsf{Hz}$, respectively, which is approximately $7.85 \cdot \mathsf{ct}$ higher than $440 \cdot \mathsf{Hz}$ or $880 \cdot \mathsf{Hz}$, respectively, compare equation (2.2) on page 61.

1.7.2. English and Dutch handbells

On the other hand, the most famous manufacturers of English handbells may be *Whitechapel Bell Foundry*[18], see [Whi13, *retrieved* May 21, 2013], which is the oldest in existence, *John Taylor & Co. Bellfounders, Bellhangers and Carillon Builders*, see [Joh13, *retrieved* May 21, 2013] and *Koninklijke klokkengieterij Petit & Fritsen*, see [Roy13, *retrieved* May 21, 2013], who ended production of handbells in 2000, see [Kea00, *Bells and Manufacturers*, *retrieved* March 10, 2005]. All three are located in Europe.

Petit & Fritsen, however, uses different tuning for their handbells. Handbells from Great Britain and the United States are tuned with an overtone of perfect twelfth (one octave and a perfect fifth) above the fundamental tone, whereas Petit & Fritsen uses a minor or major 10^{th} overtone (an octave and a minor or a major third, respectively) above fundamental, see [OnL01, *A Comparison of "Flemish" (also known as Dutch) and "English" Handbells*, *retrieved* November 20, 2007]. Some even distinguish between *English handbells* and *Dutch handbells* or *Flemish handbells* for that reason. The tuning of a Dutch handbell is similar to a usual tower bell, see [Sch04, *Petit and Fritsen (Dutch) Handbells*, *retrieved* April 4, 2013]. Hence their tune is more related to the history of handbells. Due to their harmonics, they are not used to ring massive chords.

[17] This may indicate a growing interest in handbells in East Asia and especially Japan. The request for one of their patents *JP20070173546 20070629* was in 2007.

[18] Note that Whitechapel also sells Malmark handchimes: G_4 - G_6 (2).

1.7.3. Bells on rack

Bell racks are racks with mounted handbells which are rung with a mallet and are likely used by handbell soloists or smaller professional ensembles. Handbell racks are usually used for treble bells but can also be used for bass bells requiring a different build. An example for such a custom-made rack is given by Richard Litterst, see [Kea00, *Early Leaders. Richard Litterst. Bass Bell Racks*, *retrieved* October 3, 2003].

The *Bell Matrix*™ which is offered by Schulmerich is a system that allows flexibly mounting handbells from C_5 upward. The system comes in two versions: a pillar-like rack and a horizontal board.

A set of *Cymbells*® which is offered by Malmark consists of thirteen treble bells mounted on a rack in keyboard layout. They are available as sets C_5 - C_6, C_6 - C_7 or C_7 - C_8.

1.7.4. Hybrid handbells

Some ensembles or solo ringers like to modify their handbells which might cause them to lose their warranty. An interesting variation is using a clapper from another brand. An argumentation similar to section 1.11.2 leads to the conclusion that interchanging clappers can cause serious damage to the castings. The remaining part of this paragraph is based on [EJ16]. The conversion by Malmark involves not only adjusting and changing the clapper, but modifying the casting itself. This elaborate procedure is neither required often, nor openly advertised, but applied from time to time upon request.

1.7.5. Other related products

Whitechapel manufactures *tuned cup bells*:

> "These traditional medieval style cup-shaped bells are made of the same pure bell metal as our handbells, but are tuned for their strike note only and not harmonically, with all cutting and polishing being undertaken on the outside of the casting only. They have no inside fittings and are played exclusively using suitable mallets (tapping sticks). A range of mallets is

> available. These are graded in size and weight according to type of bell. [... — M. Jedamzik].
>
> They have a beautifully clear, haunting timbre and are often played mounted on a rack. Racks may be made to order using English Oak to our simple design. Prices on application. The bells are suspended simply from their integral shank which is drilled with a hole for this purpose. [... — M. Jedamzik].
>
> They may also be supplied fitted with a leather handle, allowing them to be held and struck individually, or to be assembled as a bell tree for processional use in church.
>
> Wherever there is a call for malleted bells in music of any era, though specially suited to Early Music, Whitechapel's cup bells will create a unique impression, " [Whi13, *Handbells. Cup Bells*, *retrieved* May 21, 2013].

The Danish company *Tåsinge Kokillestøberi A/S* manufactures *aluphone*, which are cone-shaped bells made of aluminium without a clapper and pitched[19] to 442 · Hz and designed by Michael Hansen and Kai Stensgaard:

> "The Aluphone is an astonishingly versatile musical instrument capable of producing sounds normally associated with the vibraphone, the gamelan and the Tibetan singing bowl. Played with a soft mallet the instrument produces an Asiatic, meditative tone similar to the Japanese temple bell. Play the aluphone with a medium-hard mallet and it delivers the crystal clear tones associated with the vibraphone. Played with a hard mallet, the instrument will create a powerful, rich, harmonic bell sound similar to a bell," [TKA14, *Aluphone Info. The Aluphone Story*, *retrieved* April 6, 2014].

Suzuki Music USA, see [Suz13, *retrieved* May 21, 2013], located in California manufactures handchimes as part of their educational line. *Percussion Plus*, [Per13, *retrieved* November 2, 2013], located in Great Britain manufactures handchimes as part of their educational line, too.

[19] According to [SJ14], all aluphone instruments are also available to 440 · Hz pitch at request for the same price.

Belleplates Ltd., see [Bel13, *retrieved* May 21, 2013], from United Kingdom manufactures *belleplates* which are neither handbells nor handchimes. Like the name implies, the resonator which is mounted on a handle is a plate. The form of these plate roughly resembles a bell and the clapper is similar to a clapper of a handchime.

The company *Rhythm Band Instruments* manufactures *Single Ring Melody Bells®*, *KidsPlay®*, *Boomwhackers®*, and *Student Handchimes®* (in *KidsPlay®* or *Boomwhackers®* color schemes), see [Rhy14, Rhy15]. The company *Uchida®* also manufactures different models of similar bells, see [Uch16, Uch17]. All of these instruments are tuned. The clappers of the bells are mounted directly on a coil spring, thus they are not handbells as discussed in the book. Boomwhackers® are tuned percussion tubes or struck idiophones. The instruments of this paragraph are considered mostly for educational use.

1.8. Mallets

The term *mallet* is a short form of *percussion mallet*. Just to be sure: Do not use a *mallet hammer*! Mallets are essential to various ringing techniques and are widely used in handbell literature, compare table 22. Likely, the most commonly used mallet-based technique is striking a handbell resting on a table with a mallet.

Mallets are also an essential tool for *bell trees* or solo ringers.

Some ensembles replace various *pluck* techniques with corresponding mallet techniques.

Each mallet is designed for a particular range of instruments and pitches, compare table 3. The use of the appropriate mallet (for example those made by the manufacturers of handbells) is also important. An exception to this rule are Schulmerich's Greig Ashurst *Artist Series* Mallets, which were introduced on January 30, 2014, see [Sch10, *About us. News. The Greig Ashurst Artist Series Mallets are Available*, *retrieved* August 2, 2014]; The range as in table 3c is a just recommendation by Schulmerich but not a limitation, see [Sch14, page 8].

See also section 1.11.2 and section 1.11.3.

Mallets can be used as a percussion instrument, for example beating a mallet shaft with another mallet shaft simultaneously with some pairs of mallets parallel.

1.9. Chromatic scale and bell sets

The pitch notation and the nomenclature for handbell sets is given in figure 7. Note that the use of the word *octave* to determine a set of notes may differ from other parts of music theory: For example: In our small handbell world the *seventh octave* is defined as the set of handbells C_2 - $\mathsf{G}^\flat_2$ and $\mathsf{G}^\sharp_8$ - C_9 while the handbell set C_2 - C_9 is called *seven octaves of handbells*, and the handbell set C_2 - B_2 is sometimes called *the 2's*, see also figure 7.[20] The first octave consists of 13 bells. Adding $12 = 5 + 7$ bells (in total) to the bass and treble range increases the number of octaves by 1, see table 5, thus the center of a handbell set is G_5. There are, however, ensembles which do not extend in both bass and treble, since some ensemble managers deem bass handbells to contribute more to the music than treble handbells. An ensembles with handbells from C_2 to C_8 instead of handbells G_2 to G_8 still has a range of six octaves. In such a bell set, optional notes in the bass section are viewed with respect to a bell range of seven octaves, and the optional notes in the treble section are viewed with respect to a bell range of five octaves.[21] The eighth and ninth octaves have a rather academic character in the list. The staff of figure 7 might help ringers who might often change their bells and notably new ringers. The staff is reduced to C Major to gain a compact overview. One might find the circle of fifths as given in figure 8 useful.

The scientific nomenclature with the older notation B instead of H is chosen for compatibility,[22] although this may result in losing one of the most important motifs in western music, compare table 4. The table 6 gives

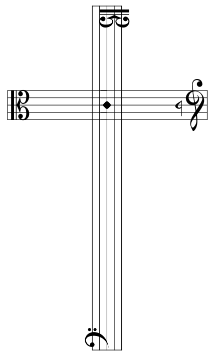
[20] The system is transferred to handchimes and other handbell sets canonically.

[21] Another point is that Malmark produces lower sixth and lower seventh octaves handchimes and lower eighth octave aluminium handbells, but no upper eighth octave handbells, neither upper seventh nor upper sixth octave handchimes.

[22] To be more precise: $\mathsf{B}=\mathsf{C}^\flat$ and $\mathsf{B}^\flat=\mathsf{A}^\sharp$. Some authors prefer to express the pitches by the symbols $\mathsf{B}^\flat$ or H respectively to avoid any confusion, neglecting that $\mathsf{H}^\flat=\mathsf{B}$ implies $\mathsf{B}^\flat=\mathsf{A}$.

a slight overview over notations and names. The fundamental tones of most bells commonly used are *equal temperament* tuned, see section 2.2.

For a human's hearing range, we refer to example 5 on page 62. Handbells are usually octavated: The concert pitch of handbells is not $440 \cdot \mathsf{Hz}$, but a compatible $880 \cdot \mathsf{Hz}$. The handbell A_4 has the fundamental frequency $440 \cdot \mathsf{Hz}$, but is written octavated, compare table 6, figure 7 and section 2.2.



The handbells below C_3 and above C_7 might be notated differently from figure 7 to avoid error-prone ledger line counting[23]. The bells C_8 - C_9 may be to ring with equivalents out of C_7 - C_8 simultaneously. Octave clefs are not usual. A common notation is octavation.

1.10. Tone of a bell

The tone of a general bell is rather complex and depends on various factors, most importantly the shape, the material and the way the bell is rung. It is possible to tune a bell to a certain pitch. For a more details see chapter 2.

1.11. Care and maintenance

Musical instruments are crafted with care and are to be handled with care. This holds for handbells and handchimes, too. Inappropriate handling or careless treatment can damage the instruments. A cracked bell can only be repaired by replacing the damaged parts. A micro crack usually cannot be seen, but several examples of a cracked handbell exist and showed a handbell out of tune.

Note that musical instruments, including handbells and handchimes are not tools. Nothing is to be placed in the castings or inside a handchime. Handchimes are not levers.

[23] As implied with figure 7, many notes which require an extensive use of ledger lines are the octavation of other notes; Thus a ringer can look for notes notated in staff or at least close to the staff.

The standard reference for care and maintenance are the manuals published by the corresponding manufacturer. Most of them are available by download of the corresponding web pages. Schulmerich has published the videos [Jun14] and Malmark's videos are available on [Mal16] or [Mal17b].

1.11.1. Handbell care and maintenance

A human's skin secretes an oily or waxy matter and sweat. These fluids are necessary for the skin but tarnish bronze and may damage handbells. The use of gloves may help to reduce the amount of sweat on the bells, see section 4.4.2. Any fingerprint or moisture should be removed from a handbell. Only a dry and clean handbell should be put back into a handbell case. A moist handbell may transfer moisture to the case interior so that even a dry handbell might get moist.

Every ringer should heed specific brand advice given by corresponding manufacturers. An example for a care and maintenance schedule is given in table 7. A handbell is a precise and precious instrument handmade by skilled artisans. Treat them (the handbell and the artisans) with respect and handle the instruments with care. Handbells can last decades and your succeeding generations of ringers should be able to make music with these fine instruments.

1.11.2. A proper use of mallets

Jacob H. Malta describes the proper use of mallets in [CMSK99, *retrieved* April 4, 2004] to gain a better sound and prevent the bell from damage:

> "Mallets should strike the bell at the same distance from the lip as does the clapper. Many people strike the bell much too far down the casting, which means that not all of the energy they put into the bell goes into the fundamental. Also, the casting is thickest at the clapper strike point, and thinnest towards the waist of the bell. So not only do you have to strike it harder to get sound, you're striking the casting at its weakest point. More bells are cracked due to improper malleting than by martellato."

The last sentence is quite surprising. Note that Jacob H. Malta sure has seen enough cracked bells to formulate such a statement.

Compare also section 1.8.

1.11.3. A proper use of martellato

Evidently the martellato technique and its derivations can damage a handbell. One has to make sure that the table is well padded and stable enough, especially when this technique is used on more handbells simultaneously. Techniques like *pluck*, *pluck lift*, *mallet on table* or *mallet lift* may be used on handbells below G_3 instead of martellato, see also [Sue07, pages 38, 60] and recall section 1.11.2 and section 5.3. [Ber12, page 69, 70, 83] even restricts martellato to handbells C_4 and upwards.

A strategy to offer additional protection for the martellato technique is to use two different layers of foam for the tables' pads: One layer of soft foam (top) to actually damp the handbells and allow the bells to sink in the foam to secure their positions and one layer of harder foam (below) to offer more resistance for the martellato technique. The harder layer is just in case the ringer accidentally uses too much force.

1.11.4. Handchime care and maintenance

Handchimes are made of coated metal and the surface is, therefore, more durable than a handbell's bronze casting. However handchimes can be bent unintentionally. Schulmerich therefore points out in [Sch07, page 2] that the martellato technique on handchimes is a misuse of the instrument and will likely damage the instruments. Schulmerich similarly emphasizes that a chime should not be rung with force.

1.11.5. One note on cases with wheels

Large handbell cases filled with handbells are usually heavy, so Schulmerich (*Ring 'N' Roll cases*), Malmark, *Port-A-Bell*® and *The Ultimate Bell & Chime Case*™ offer cases with wheels, the latter two being third party for Malmark and Schulmerich. There is no doubt that these cases make transporting handbells more comfortable and likely healthier for ringers. Note that there are still unsuitable terrains for wheels, for example stairways.

Cases with wheels may have to be carried the old way occasionally. Always remember: "Do not be upset when you have to carry your cases with wheels; be happy when you can use the wheels!"

1.12. Handbell organizations

There are some handbell organizations which want to represent the ringers of their country. Although they focus on representing the ringers of their nation, they welcome ringers from abroad for sure. Usually individual members can join as well as ensembles. These organizations are:

Asia International Handbell Association (Hong Kong) (AIHA) founded in 2008. For more information see [AIHA08].

American Guild of English Handbell Ringers (AGEHR) see *Handbell Musicians of America (HMA).*

Handbell Association of Hong Kong (HAHK) founded in 2006. For more information see [HAHK06].

Handbell Guilds of Canada (HGC) is the parent organization of provincial associations of Canada. The *Alberta Guild of English Handbell Ringers* is oldest of these and was founded in 1983.

Handbell Musicians of America (HMA) founded in 1954 and renamed from *American Guild of English Handbell Ringers* on October 1, 2011, see [Pot11, page 1]. This likely is the most important and most influential handbell organization. Many famous persons of the international handbell community are members of the HMA. For more information see [HMA14].

Handbell Ringers of Great Britain (HRGB) founded in 1967. For more information see [HRGB14].

Handbell Ringers of Japan (HRJ) founded in 1976. For more information see [HRJ13].

Handbell Ringers of Singapore (HRSG) was founded before 2013, compare [Sen12, *Meet the Team*, *retrieved* August 16, 2016]

Handbell Society of Australasia (HSA) founded in 1983. For Australia and New Zealand. For more information see [HSA13]

Korean Handbell Association (KHA) founded in 1985, compare [Kea00, *Frequently Asked Questions: Other Guilds*, *retrieved* August 16, 2004]. For the *Republic of Korea* (South Korea).

Some of these organizations have (local) subgroups. One of these organizations has hosted the *International Handbell Symposium* every two years since 1984. The *International Handbell Committee (IHC)* was founded in 1990 and consists of heads of the above listed organizations, [IHC12, *About*, *retrieved* May 28, 2013]:

> "The Mission Statement of the International Handbell Committee is the promotion of the art of handbell ringing throughout the world. We, the International Handbell Committee, share a passion for the love of music expressed through handbells. As the International Committee dedicated to this art, we articulate our vision for the future to promote:
>
> 1. communication between nations,
> 2. handbells as a musical art,
> 3. world peace through the spirit of music."

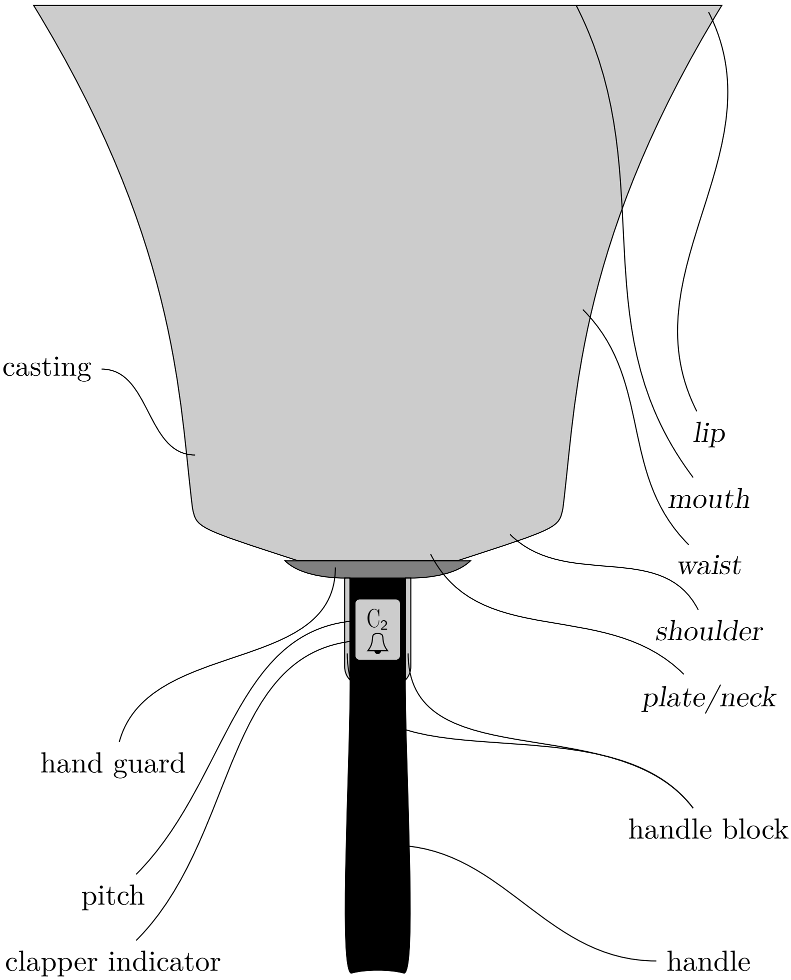
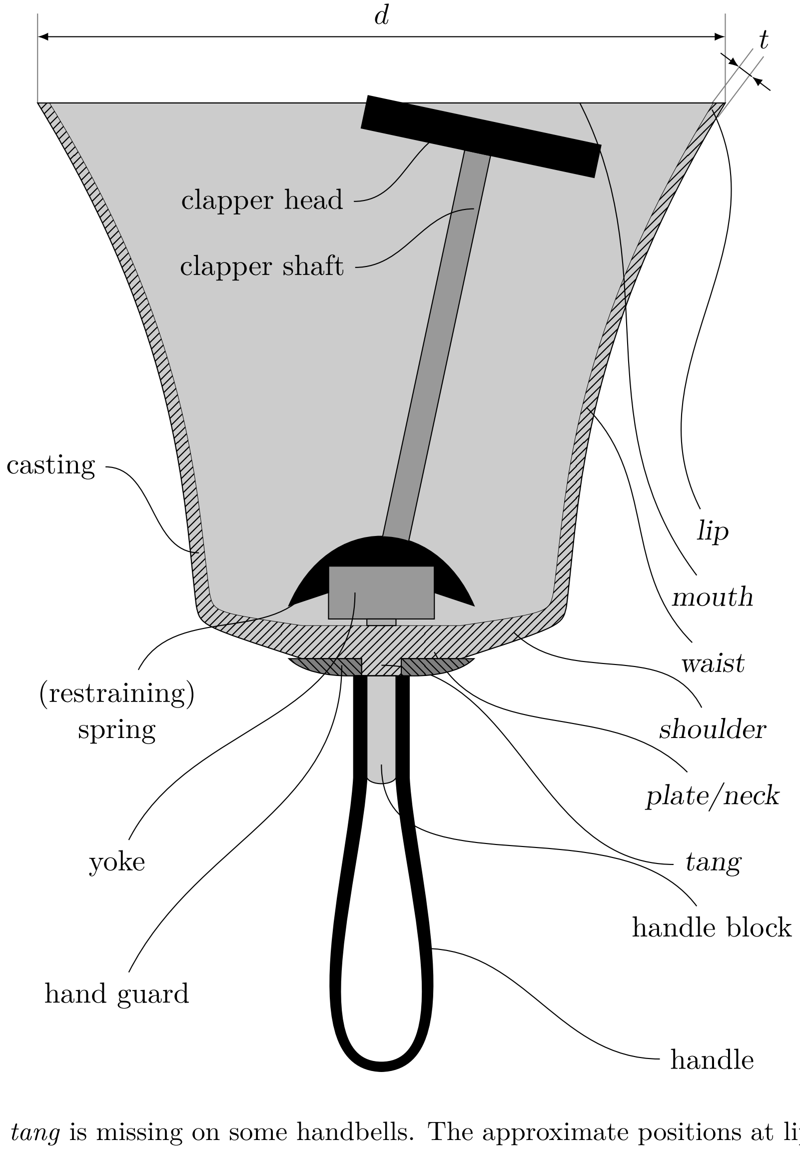


Figure 1: Frontal view of a handbell (schema)



The *tang* is missing on some handbells. The approximate positions at lip for the variables t and d are drawn for the mode $[2, 0]$, see section 2.6.1.

Figure 2: Side view (cross section) of a handbell (schema)

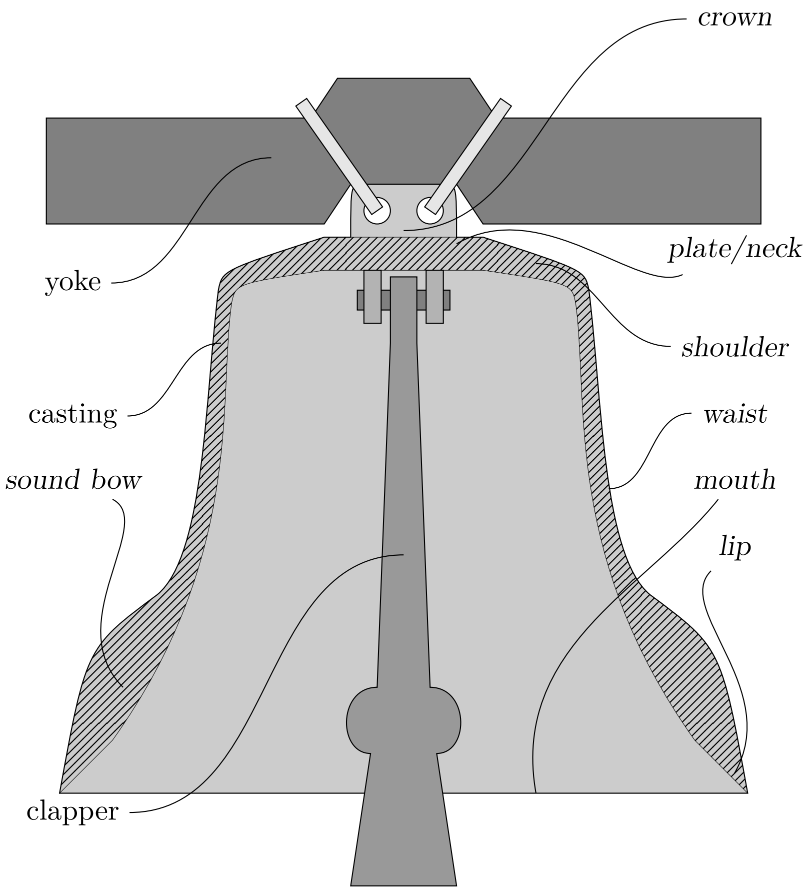
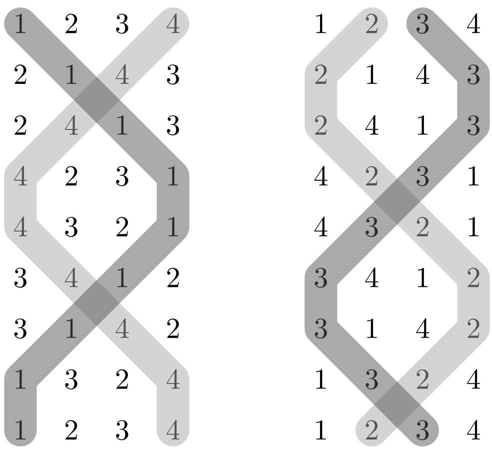


Figure 3: Cross section of a church bell (schema)



(a) blue lines for bells 1 and 4 highlighted

(b) blue lines for bells 2 and 3 highlighted

Note that each *change* or *peal* differs from the previous one by swapping either positions $1, 2$ and positions $3, 4$ or by swapping positions $2, 3$. Note the resulting symmetry of the *blue lines* for 1 and 4 respectively 2 and 3. This specific bell change sequence is called *plain hunt* for four bells.

Figure 4: An example for change ringing with four bells ($n = 4$)

n	1	2	3	4	5	6
$\#\mathfrak{S}_n$	1	2	6	24	120	720
$n\#\mathfrak{S}_n$	1	4	18	96	600	4 320
time	$1 \cdot \mathsf{s}$	$4 \cdot \mathsf{s}$	$18 \cdot \mathsf{s}$	$96 \cdot \mathsf{s}$	$10 \cdot \mathsf{min}$	$72 \cdot \mathsf{min}$

n	7	8	9	10
$\#\mathfrak{S}_n$	5 040	40 320	362 880	3 628 800
$n\#\mathfrak{S}_n$	35 280	322 560	3 265 920	36 288 000
time	$9.8 \cdot \mathsf{h}$	$9.8 \cdot \mathsf{d}$	$37.8 \cdot \mathsf{d}$	$1.1 \cdot \mathsf{a}$

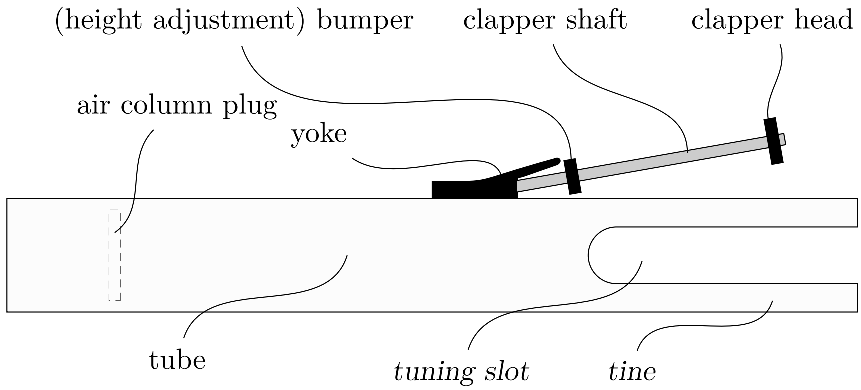
n	11	12	13
$\#\mathfrak{S}_n$	39 916 800	479 001 600	6 227 020 800
$n\#\mathfrak{S}_n$	439 084 800	5 748 019 200	80 951 270 400
time	$13.9 \cdot \mathsf{a}$	$182.1 \cdot \mathsf{a}$	$2565.1 \cdot \mathsf{a}$

The first row represents the number of bells. The second row represents the number of different peals, recall $n! = \#\mathfrak{S}_n$ for $n > 0$. The third row represents the number of bells rung when ringing all peals. The fourth row represents the the growth of the third row: The last row shows the approximate time needed to ring all peals when each ring would take one second. Note $1 \cdot \mathsf{a} = 365.25 \cdot \mathsf{d}$.

Table 1: Number of permutations

In this picture the top corresponds to the north. The *Pfarrhaus* (north; former parsonage, now community center), the graveyard (center), the Lutheran *St. Nicolai church* (east), and the fire department (south), photography by MORITZ BUHR on April 10, 2016, postproduction by MICHAEL JEDAMZIK.

Figure 5: St. Nicolai Church in Wiedensahl



The *air column plug*, which is not visible from this perspective but indicated in this figure, is mounted inside the tube and has an influence on the pitch, hence it should not be moved.

Figure 6: Side view of a handchime (schema)

manufacturer	Schulmerich	Malmark	Prima Gakki
bronze handbell	C_2 - C_9 (7)	C_2 - C_9 (7)	C_4 - C_7 (3)
aluminium handbell	none	G_1 - $\mathsf{G}^{\flat}_3$ (2)	none
brass handbell	$\mathcal{C}_5$ - $\mathcal{C}_7$ (2)	none	none
handchime	c_3 - c_8 (5)	c_2 - c_8 (6)	none

(a) American handbells

manufacturer	Whitechapel	Taylor	Petit & Fritsen
bronze handbell	G_2 - C_8 (5.5)	C_3 - C_8 (5)	$\mathbb{C}_3$ - $\mathbb{C}_8$ (5)

(b) English and Dutch handbells

product	**manufacturer**	**bell range**	
handchime	Suzuki	c_3 - g_7	(4.5)
handchime	Percussion Plus	c_4 - c_7	(3)
belleplate	Belleplates	c_2 - c_8	(6)
tuned cup bell	Whitechapel	$\mathfrak{c}_4$ - $\mathfrak{c}_8$	(4)
aluphone	Aluphone	$\mathfrak{c}_4$ - $\mathfrak{c}_7$	(3)

(c) related products

With a length of $170 \cdot \mathsf{cm} \approx 67 \cdot \mathsf{inch}$ (in case of their C_2) the lower handchimes made by Malmark are not really *hand*chimes. These chimes are usually rung on a rack with mallets.

Table 2: Manufacturers overview

color	range
Plum	C_2 - $F^\sharp_3$
Burgundy	G_2 - B_3
Green	F_3 - G_4
Jade	F_4 - B_4
Tan	G_4 - G_5
Light Blue	E_5 - D_6
Brown Rubber	C_6 - A_6
White Poly	G_6 - C_8
Black Rubber	C_5 - G_7
White Nylon	C_7 - G_8

(a) Jeffers *TruTimbre®*

color	range
Lambswool	aluminium handbell, C_2 - B_2, c_2 - b_2
Large Yellow	C_3 - $F^\sharp_3$, g_2 - $f^\sharp_3$
Navy Blue	G_3 - B_3
Small Yellow	C_4 - $F^\sharp_4$
Light Blue	G_4 - E_5
Red	F_5 - $D^\sharp_6$
Gray	E_6 - C_9
Clear	"Special Applications for Brighter Treble"

(b) Malmark

color	range
Forest Green Yarn	C_2 - E_3
Raspberry Yarn	G_2 - B_3
Black Poly	G_3 - G_4
Red Poly	C_4 - C_5
Silver Poly	G_4 - G_5
Blue Poly	C_5 - E_6
Blue Solid	G_5 - C_7
Black Solid	G_6 - C_9

(c) Schulmerich GREIG ASHURST *Artist Series*

color	range
Brick Red	C_2 - A_2
Purple	G_2 - $F^\sharp_3$
Dark Blue	G_3 - E_4
Green	F_4 - B_4
Gold	C_5 - $A^\sharp_5$
Pink	B_5 - G_6
Blue Hard Rubber	$G^\sharp_6$ - C_8
White Nylon	$C^\sharp_8$ - C_9

(d) Schulmerich *MasterTone*

Referecences: [Jef10, *Equip. & Supplies. Mallets*, *retrieved* Februar 16, 2017], [Mal14, page 5], [Mal17a, page 8], [Mal13a, *Shop online. Accessories. Mallets. MH5 Lambswool Mallet G1-B2 Aluminum*, *retrieved* Februar 16, 2017], [Sch13, page 4], and [Sch14, page 8].

Table 3: Manufacturers overview (mallets)

English	$C^{\flat}$	C	$C^{\sharp}$	$D^{\flat}$	D	$D^{\sharp}$	$E^{\flat}$	E	$E^{\sharp}$	$F^{\flat}$	F	$F^{\sharp}$
German	Ces	C	Cis	Des	D	Dis	Es	E	Eis	Fes	F	Fis
English	$G^{\flat}$	G	$G^{\sharp}$	$A^{\flat}$	A	$A^{\sharp}$	$B^{\flat}$	B	$B^{\sharp}$			
German	Ges	G	Gis	As	A	Ais	B	H	His			

The German keys *Eïs* and *Aïs* do not contain a diphthong but a diaeresis.

Table 4: Comparison of key signature names

octave set	1	2	3	4	5	6	7	7.5	8	9
lowest bell	C_5	G_4	C_4	G_3	C_3	G_2	C_2	G_1	G_1	C_1
highest bell	C_6	G_6	C_7	G_7	C_8	G_8	C_9	C_9	G_9	C_{10}
number of bells	13	25	37	49	61	73	85	90	97	109

Table 5: Octave sets

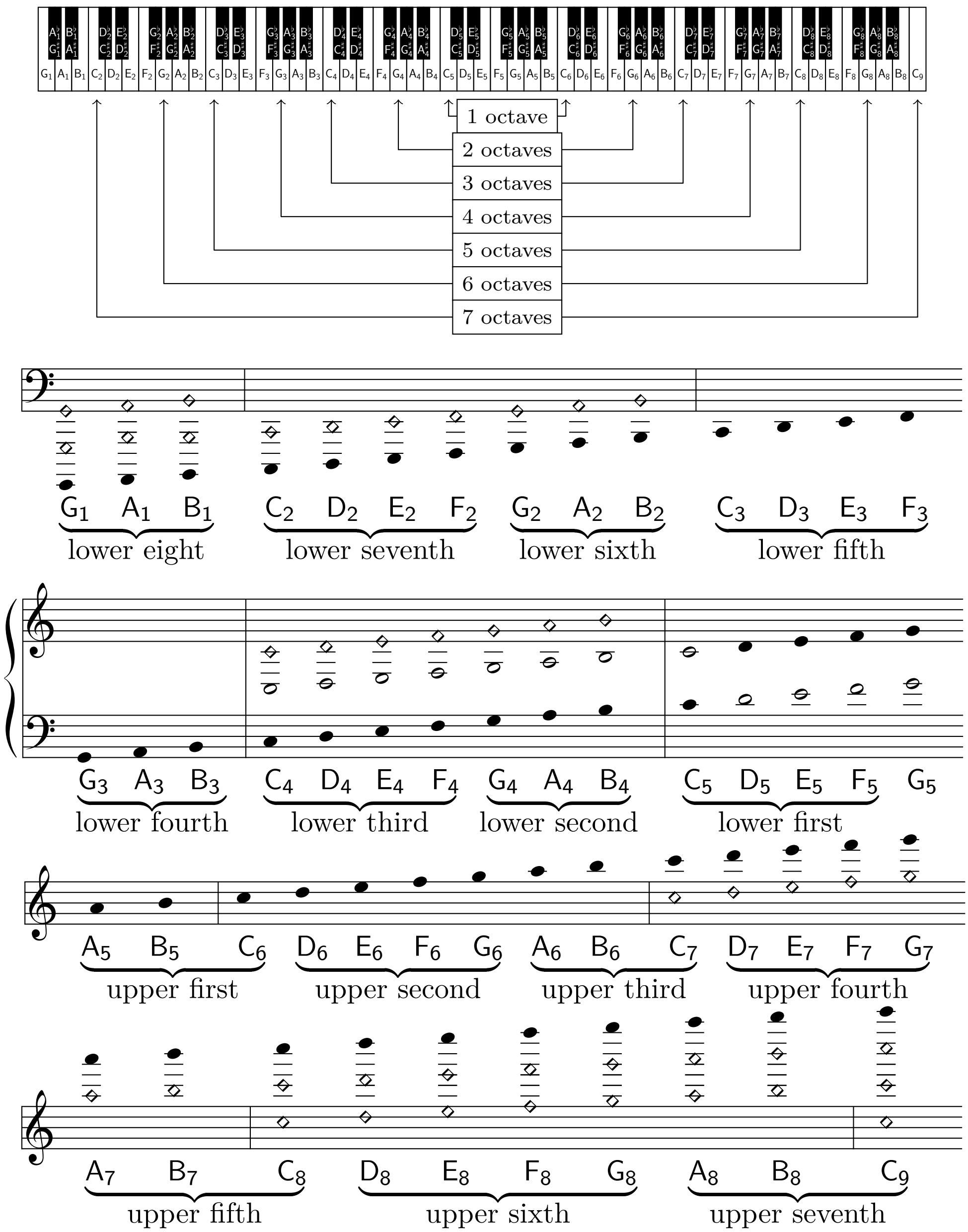
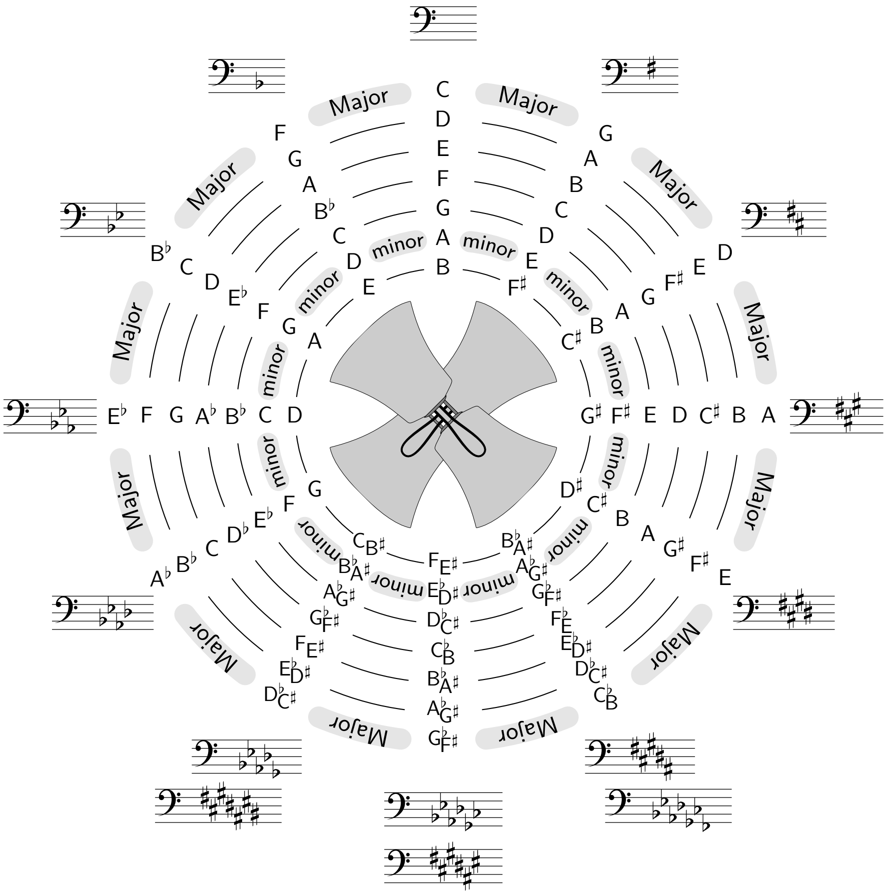


Figure 7: Pitch notation and bell sets



The tones in the grey circles show the names of the scales, for example, the F minor scale consists of the tones F, G, A♭, B♭, C, D♭, E♭,F.

Figure 8: Circle of fifths

name (G)	...	subsubkontra A	subkontra A	kontra A
name (E)	...	double contra A	sub contra A	contra A
pitch (eq. tmp)	...	13.75 · Hz	27.5 · Hz	55 · Hz
bilinear	...	−tA	−uA	−wA
pattern	...	52A	45A	38A
Helmholtz	...	,,,A *or* A,,,	,,A *or* A,,	,A *or* A,
scientific	...	A_{-1}	A_0	A_1
name (G)	großes A	kleines a	eingestrichenes a	zweigestrichenes a
name (E)	great A	small a	one-lined a	two-lined a
pitch (eq. tmp)	110 · Hz	220 · Hz	440 · Hz	880 · Hz
bilinear	−xA	−xA	−yA	zA
pattern	31A	24A	17A	10A
Helmholtz	A	a	a'	a''
scientific	A_2	A_3	A_4	A_5
name (G)	dreigestrichenes a	viergestrichenes a	fünfgestrichenes a	...
name (E)	three-lined a	four-lined a a	five-lined a	...
pitch (eq. tmp)	1760 · Hz	3520 · Hz	7040 · Hz	...
bilinear	yA	xA	wA	...
pattern	3A	03A	003A	...
Helmholtz	a'''	a''''	a'''''	...
scientific	A_6	A_7	A_8	...

Table 6: Handbell pitch notation overview

schedule	**application**
after use	Clean the casting with polishing cloth. Use jewelers rouge (wiping it off completely afterwards) if necessary. Report any changes or malfunctions to the director.
monthly	Check all screws, adjust if necessary. Inspect all parts.
half-yearly	Clean yoke assembly. Check the clapper head and spring adjustments. Polish the casting with polish if necessary.
yearly	Clean all non-metal parts of each instrument. Inspect and repair cases. Vacuum all cases. Remove all bells from cases and let them air out for two days.

This schedule is based on [Sch11, pages 10, 11]. Schulmerich notes that the schedule may need to be adjusted based on individual needs. See the care and maintenance manuals of the corresponding manufacturers for more details.

Table 7: Handbell care schedule

Pagina vacat.

Mvsica est exercitivm arithmeticae occvltvm nescientis se nvmerare animi.

from letter to CHRISTIAN GOLDBACH,
April 17, 1712
GOTTFRIED WILHELM VON LEIBNIZ

2. Short introduction to related physics

This chapter is dedicated to selected physical aspects.

Every advanced musician needs some knowledge about physics. The history of pitch which dates back in ancient Greek is not discussed here. The physics are treated as given and physical models as given here are simplified. We neglect inharmonicity and stretched tuning in this chapter, compare [Hin12]. The mathematics however are rather simple.

The sections 2.1 and 2.2 are just reminders and introduce important background.

This chapter uses mathematical jargon. For example the word "iff" means "if and only if". We write the imaginary unit as i, hence $\mathbb{C} = \mathbb{R} + i\,\mathbb{R}$, the projections to the real part and imaginary part as $\Re$ or $\Im$, respectively, the *logarithmus naturalis* as log, and we define ls as the logarithm of the base $\sqrt[12]{2}$, note

$$\operatorname{ls} x = \frac{12}{\log 2} \log x$$

for all $x > 0$. In the latter half, we will write the one-dimensional sphere as

$$S^1 \cong T^1 \cong \mathbb{R}/\mathbb{Z} \cong \mathbb{R}/(2\pi\,\mathbb{Z}),$$

the n-dimensional Euclidean[24] space as $\mathbb{E}^n$, its norm by $|\cdot|$, its inner product by $\langle\ \cdot\ ,\ \cdot\ \rangle$, the norm of a normed vector space B by $\|\cdot\|_B$ and the inner product of a pre-Hilbert[25] space H by $\langle\ \cdot\ ,\ \cdot\ \rangle_H$, use Ricci[26] calculus, and write the Riemannian Hodge[27] star operator on differential forms as $*$. For simplicity we use the abbreviations $L^p(M)$ and $W^{k,p}(M)$, respectively, for the function spaces $L^p(M,g)$ and $W^{k,p}(M,g)$ on a Riemannian manifold (M,g) whenever the choice of metric is clear by context. Some of our calculations will require unit conversions:

$$\begin{aligned}
1\cdot\mathsf{mile} &= 1760\cdot\mathsf{yard} = 1609.344\cdot\mathsf{m},\\
1\cdot\mathsf{yard} &= 3\cdot\mathsf{foot} = 0.9144\cdot\mathsf{m},\\
1\cdot\mathsf{foot} &= 12\cdot\mathsf{inch} = 0.3048\cdot\mathsf{m},\\
1\cdot\mathsf{inch} &= 25.4\cdot\mathsf{mm} = 0.0254\cdot\mathsf{m},\\
1\cdot\mathsf{pt}_{\mathrm{\TeX}} &= \frac{1}{72.27}\cdot\mathsf{inch} = \frac{2540}{7227}\cdot\mathsf{mm},\\
1\cdot\mathsf{p} &= 22633485174381732 16473\cdot\mathsf{ym},\\
1\cdot\mathsf{lb_m} &= 0.45359237\cdot\mathsf{kg}.
\end{aligned}$$

2.1. Tone of an instrument

A sounding instrument vibrates and these oscillations are emitted to the surrounding air and this oscillation is transferred to our ears. The instruments vibrate *periodically*: The pattern of back and forth movements repeat, neglecting sound decay. An important physical quantity is the *frequency* which is the number of periods per second. The oscillations do not need to be sine-like.

There are pairs of different instruments which can play the same pitch, but their sound is notably different. This difference is caused by their

[24]Euclidean geometry is named after the Alexandrian Greek mathematician Εὐκλείδης (Euclid of Alexandria).

[25]Hilbert and pre-Hilbert spaces are named after the German mathematician David Hilbert.

[26]The various aspects of Ricci calculus are named after the Italian mathematician Gregorio Ricci-Curbastro.

[27]The Hodge dual is named after the Scottish mathematician William Vallance Douglas Hodge.

overtones. A tone of an instrument is decomposed into several oscillations of several frequencies. These oscillations add to one characteristic oscillation, which characterizes the *timbre* of an instrument. This principle is called *superposition* and is illustrated in figure 9.

The oscillation with the lowest frequency is named *fundamental tone*. This frequency defines the pitch and is the most intense. Table 6 gives some examples. Oscillations and frequencies are often identified for convenience.

For example, a piano and an organ can play at the same (fundamental) tone, but sound notably differently due to their characteristic overtones.[28] An investigation on a single church bell in [Ros84, pages 112-132] classifies 134 partials which does not mean that these are all.

Another important physical quantity is the *amplitude* of an oscillation: The amplitude corresponds roughly to the volume of the oscillation. There are different notions of amplitude. The *peak amplitude* is the height of the peak of the oscillation. To give a more precise one: Let $f\colon D \to \mathbb{R}$ be an oscillation, then the peak amplitude is defined by its supremum norm

$$\|f\|_{L^\infty(D)}.$$

2.2. Equal tempered pitch

Handbells, like many other instruments, are *equal temperament* tuned. This implies that frequencies of two bells differ by factor 2, iff they are octavated, see table 6. To be more precise:

Definition 1. A tone with (fundamental) frequency f_1 is exactly one *semitone*[29] (m_2) lower than a tone of (fundamental) frequency f_2 iff

$$\sqrt[12]{2} \cdot f_1 = f_2.$$

One *tone* equals two semitones. One *cent* (ct) is one-hundredth of a semitone (linear scaled): $1 \cdot \mathsf{m}_2 = 100 \cdot \mathsf{ct}$.

[28] We now know, what the iconic title *Overtones* of the official journal of the Handbell Musicians of America means. Mono-pitched instruments do not play an important role.

[29] Its symbol is an abbreviation for the interval *minor second*.

The approximation $\sqrt[12]{2} \approx 1.059$ thus implies that the higher bell has about 6% more oscillations. Per induction: The fundamental tone f_1 is exactly $k \cdot \mathsf{m}_2$ higher than the fundamental frequency f_2 of another handbell iff

$$f_1 = \sqrt[12]{2}^k \cdot f_2 = 2^{k/12} \cdot f_2 = f_2 \exp \frac{k}{\operatorname{ls} e}. \tag{2.1}$$

This notion can be extended trivially to any $k \in \mathbb{R}$. Since $f_1 = 0$ or $f_2 = 0$ would imply that one of them is not really an oscillation as there are no movements and thus no tone at all, such a $k \in \mathbb{R}$ as in (2.1) exists for any two frequencies, hence (2.1) is equivalent to

$$k = \operatorname{ls} f_1 - \operatorname{ls} f_2 = \operatorname{ls} \frac{f_1}{\mathsf{Hz}} - \operatorname{ls} \frac{f_2}{\mathsf{Hz}}.$$

One octave equals a difference of $12 \cdot \mathsf{m}_2$, since $2^{12/12} = 2$.

To make the principle clear, we show the results of some calculations which are usually done by a calculator, figure 7, and probably table 8:

Example 2. These examples are based on the fundamental tone of A_2, compare table 6: The fundamental tone of handbell of pitch $\mathsf{A}_2^\sharp$ is exactly

$$110 \sqrt[12]{2} \cdot \mathsf{Hz} \approx 116.541 \cdot \mathsf{Hz},$$

the fundamental pitch of a F_2 is exactly $4 \cdot \mathsf{m}_2$ lower:

$$110 \sqrt[12]{2}^{-4} \cdot \mathsf{Hz} = 55 \sqrt[3]{2}^2 \cdot \mathsf{Hz} \approx 87.307 \cdot \mathsf{Hz},$$

the fundamental pitch of a C_2 is exactly $9 \cdot \mathsf{m}_2$ lower:

$$110 \sqrt[12]{2}^{-9} \cdot \mathsf{Hz} = 55 \sqrt[4]{2} \cdot \mathsf{Hz} \approx 65.406 \cdot \mathsf{Hz},$$

and the fundamental tone of a A_3 is $12 \cdot \mathsf{m}_2$ higher:

$$110 \sqrt[12]{2}^{12} \cdot \mathsf{Hz} = 2 \cdot 110 \cdot \mathsf{Hz} = 220 \cdot \mathsf{Hz}.$$

The definition of differences by semitones is a linearization of the exponential scale of frequencies and meets a human's auditory perception. This concept itself is similar to the *pondus Hydrogenii* (*pH-value*), *sound pressure* (loudness measured in bel or dB, respectively) or practically *exposure value* from photography, although the scales themselves are not logarithmic. Compare the scales of f-number or film speed, the later usually measured in ISO.

The following examples are more interesting for those who are already familiar with some aspects of tuning.

Example 3. The *Pythagorean comma*

$$\operatorname{ls}\left(\left(\frac{3}{2}\right)^{12} \cdot 2^{-7}\right) \cdot \mathsf{m}_2 = 12(\operatorname{ls} 3 - 19) \cdot \mathsf{m}_2 \approx 23.4600 \cdot \mathsf{ct},$$

causes the infamous *wolf interval* in Pythagorean[30] tuning. The factor $3/2$ corresponds to a perfect fifth in Pythagorean tuning. The comma

$$\operatorname{ls}\left(\left(\left(\frac{3}{2}\right)^{6} \cdot 2^{-3}\right) \cdot \sqrt[12]{2}^{-6}\right) \cdot \mathsf{m}_2 = -\operatorname{ls}\left(\left(\left(\frac{2}{3}\right)^{6} \cdot 2^{4}\right) \cdot \sqrt[12]{2}^{-6}\right) \cdot \mathsf{m}_2$$

is exactly half the Pythagorean comma and causes the similar and infamous *diabolus in musica* in Pythagorean tuning. Both comma vanish in the equal temperament tuning, hence the effects are nonexisting.

Example 4. There are numerous conventions of a *concert pitch*. Many of them refer to A_4. We just want to compare the $442 \cdot \mathsf{Hz}$ and $884 \cdot \mathsf{Hz}$:

$$(\operatorname{ls} 884 - \operatorname{ls} 880) \cdot \mathsf{m}_2 = (\operatorname{ls} 442 - \operatorname{ls} 440) \cdot \mathsf{m}_2 = 7.8514 \cdot \mathsf{ct} \tag{2.2}$$

Finally calculations for the lovers of historically informed baroque performances:

$$440\,\sqrt[12]{2}^{-1} \cdot \mathsf{Hz} \approx 415.3047 \cdot \mathsf{Hz},$$
$$\left(\operatorname{ls}\frac{440}{\sqrt[12]{2}} - \operatorname{ls} 415\right) \cdot \mathsf{m}_2 \approx 1.2706 \cdot \mathsf{ct}.$$

[30] This temperament is attributed to the Greek mathematician Πυθαγόρας ὁ Σάμιος (Pythagoras of Samos).

2.3. Selected aspects of psychoacoustics

In this chapter 2 we will have a look on some aspects of physical acoustics. There are some other relevant and rather complex effects of psychoacoustics which we do not cover here since they cannot be explained by mathematics. We refer to [HA09] or [ZF99]. But we should have a quick look on a small selection on psychoacoustics in examples 5 and 6 and remark 7.

Example 5. Humans usually are able to hear tones in a range of $20 \cdot \mathsf{Hz}$ to $20000 \cdot \mathsf{Hz}$, compare [HA09, page 89]. This corresponds approximately to the pitch $\mathsf{D}_0^\sharp$ or $\mathsf{D}_9^\sharp$, respectively, since

$$\begin{aligned} \operatorname{ls} 20000 - \operatorname{ls} 440 &\approx 5 \cdot 12 + 6.076, \\ \operatorname{ls} 440 - \operatorname{ls} 20 &\approx 4 \cdot 12 + 5.513, \end{aligned}$$

which sums up to approximately 9 octaves and $11.589 \cdot \mathsf{m}_2$.

Example 6. If we hear two tones with an interval of a perfect fifth like C_2 and G_2, we may be able to hear another tone whose frequency is given as the difference of the two tones, which means this imaginary tone is

$$-\operatorname{ls}\left(\sqrt[12]{2}^{7} - 1\right) \cdot \mathsf{m}_2 \approx 12.059 \cdot \mathsf{m}_2$$

lower, compare [HA09, section 5.5.4]. That means in addition to C_2 and G_2, we would hear the *combination tone* or *resultant* which is almost a C_1. This is not to be confused with example 19.

Remark 7. The strike note of a bell in general is more complicated and subjective, compare [FR98, page 682]. To make this even more complicate, the notion of the *fundamental* of church bells and handbells differs, compare [FR98, table 21.1, section 21.8].

2.4. The Doppler effect

The well known *Doppler effect*[31] causes change of audible frequencies. This effect is known from vehicle mounted sirens (for example on fire trucks)

[31] This phenomenon which is also known as *Doppler shift* is named after the Austrian physicist Christian Doppler.

or real church bells and even occurs in handbells considering that a usual auditor almost does not move his head while a sounding handbell may move (in a circle).

2.4.1. Formula

Let us denote[32] the frequency of a handbell by f_b with $f_b \cdot \mathsf{s} > 0$, the frequency of the handbell as heard by the auditor as f_a with $f_a \cdot \mathsf{s} > 0$, the speed of sound by c and the velocity of the bell moving towards the auditor by v with $c \cdot \mathsf{s} \cdot \mathsf{m}^{-1} > |v| \cdot \mathsf{s} \cdot \mathsf{m}^{-1} \geq 0$. Then the formula

$$f_a = \frac{c}{c-v} f_b =: \sqrt[12]{2}^{\frac{T(v)}{\mathsf{m}_2}} \cdot f_b \tag{2.3}$$

is well known from classical physics. Since the ratio of two consecutive semitones is simply $\sqrt[12]{2}$, the increase of pitch $T(v)$ caused by the Doppler effect is obtained by solving the right equation (2.3):

Proposition 8. *Let c be the speed of sound and v the velocity of a bell moving directly towards the non-moving auditor with $c \cdot \mathsf{s} \cdot \mathsf{m}^{-1} > |v| \cdot \mathsf{s} \cdot \mathsf{m}^{-1}$. Then the difference of pitch heard by the auditor is given by*

$$T(v) = 100\Big(\operatorname{ls} c - \operatorname{ls}(c-v)\Big) \cdot \mathsf{ct}. \tag{2.4}$$

We see from (2.4) that $T(v) = 0 \Leftrightarrow v = 0$ and that $T(v) \cdot \mathsf{ct}^{-1}$ is strictly increasing, since ls is a strictly increasing function and T is independent from f_b, compare (2.3). We also observe that T is independent from the pitch and also independent from the units used, as expected.

We have (in dry air at $20°\mathsf{C} = 86°\mathsf{F}$) $c \approx 343.2 \cdot \mathsf{m} \cdot \mathsf{s}^{-1}$. We note

$$T\Big(\pm 0.99 \cdot \mathsf{m} \cdot \mathsf{s}^{-1}\Big) \approx \pm 5.00 \cdot \mathsf{ct},$$

which is barely distinguishable by the human ear in perfect conditions and

$$T\Big(-10.0565 \cdot \mathsf{m} \cdot \mathsf{s}^{-1}\Big) \approx -50.000 \cdot \mathsf{ct},$$
$$T\Big(9.7703 \cdot \mathsf{m} \cdot \mathsf{s}^{-1}\Big) \approx 50.000 \cdot \mathsf{ct},$$

which is half a semitone, see [Loe06, section 1.2.4].

[32] In section 2.4 we consequently use units.

2.4.2. Back-of-the-envelope calculations

We do some simple calculations to get an idea of the Doppler effect.

Example 9 (Shake technique). Concerning one ringer doing a shake as follows: He shakes the handbell back and forth 6 times per second. The bell moves $10.16 \cdot \mathsf{cm} = 4 \cdot \mathsf{inch}$ from one side to the other. Thus the overall velocity of the bell is $v_1 = (2 \cdot 10.16 \cdot \mathsf{cm}) \cdot 6 \cdot \mathsf{Hz} = 1.2192 \cdot \mathsf{m} \cdot \mathsf{s}^{-1}$ and therefore $T(v_1) \approx 6.1610 \cdot \mathsf{ct}$ and $T(-v_1) \approx -6.1392 \cdot \mathsf{ct}$. The difference between these two values is approximately $0.123 \cdot \mathsf{m}_2$.

Example 10 (Ringing without damping). Concerning one ringer ringing a bell again (without damping) by just pushing the bell forward: He can pull the handbell back and forth 4 times per second. The bell moves $60.96 \cdot \mathsf{cm} = 24 \cdot \mathsf{inch}$ from a position next to his torso and away from him. Thus the overall velocity of the bell is $v_2 = (2 \cdot 60.96 \cdot \mathsf{cm}) \cdot 4 \cdot \mathsf{Hz} = 4.8768 \cdot \mathsf{m} \cdot \mathsf{s}^{-1}$ and therefore $T(v_2) \approx 24.7769 \cdot \mathsf{ct}$ and $T(-v_2) \approx -24.4273 \cdot \mathsf{ct}$. The difference between these two values is approximately $0.492 \cdot \mathsf{m}_2$.

Example 11 (Swing technique). One physical effect of the swing technique (see section 5.3.1) is the fact that (using a simple model) the bell emits sound parallel to the hand guard of the bell which leads to an increasing and decreasing volume depending on the movement. This effect is not treated in this section. What follows are calculations for the Doppler effect.

We use a two-dimensional model: Let the position on a back swing of the bell be parameterized by the curve

$$\begin{aligned} \gamma \colon \left[0, {}^1\!/\!_l\right] &\to \mathbb{R}^2 \\ t &\mapsto \begin{pmatrix} p + r\cos(+\pi l t) \\ q + r\sin(-\pi l t) \end{pmatrix}, \end{aligned}$$

where $r > 0$ is the radius of the circle and $l \neq 0$ is the (signed) number of circulations per second, and (p, q) with $|p| > r$ is the position of the shoulder of the ringer, and let the position of the non-moving auditor be in the origin 0.

The movements of the bell require the general Doppler equation

$$f_a = \frac{c}{c - \left\langle \dot{\gamma}, \frac{0-\gamma}{|0-\gamma|} \right\rangle} f_b, =: \sqrt[12]{2}^{\frac{T_3}{\mathsf{m}_2}} \cdot f_b \tag{2.5}$$

from which (2.3) follows, hence the change of pitch is exactly

$$T_3 = 100 \left(\operatorname{ls} c - \operatorname{ls} \left(c + \frac{\langle \dot{\gamma}, \gamma \rangle}{|\gamma|} \right) \right) \cdot \mathsf{ct}.$$

Note: The argument of T_3 is time t, not the velocity vector $\dot{\gamma}(t)$.

We set $r = q = 20 \cdot \mathsf{inch} = 0.508 \cdot \mathsf{m}, p = 37 \cdot \mathsf{foot} = 11.2776 \cdot \mathsf{m}$ and $l = 1$. Numeric calculations show that the maximum and minimum of $T_3|_{[-1/l, 1/l]}$ are approximately $8.0693 \cdot \mathsf{ct}$ or $-8.0318 \cdot \mathsf{ct}$, respectively. The difference between these two values is approximately $0.161 \cdot \mathsf{m}_2$.

2.4.3. Conclusion

The change of pitch as given in the previous examples may be upper bounds of the techniques. They were measured in experiments without handbells, hence the real velocities are likely smaller.

In perfect conditions the discussed ringing techniques change the frequency notably. It is questionable if the Doppler effect, when having only a small influence on the pitch, is really noticeable in concert-situation since other bells may have an influence, as well as the fact that a usual auditor does not focus on the changing pitch of a specific single bell, but rather on the music as a whole.

We might look at an example almost everybody is familiar with for comparison:

Example 12 (Fire truck). We assume that a fire truck with a turned on siren is driving towards[33] us with a velocity of $v = 20 \cdot \mathsf{m} \cdot \mathsf{s}^{-1} = 72 \cdot \mathsf{km} \cdot \mathsf{hour}^{-1} \approx 44.74 \cdot \mathsf{mile} \cdot \mathsf{hour}^{-1}$. Hence the change of pitch is $T(v) \approx 103.9468 \cdot \mathsf{ct}, T(-v) \approx -98.0576 \cdot \mathsf{ct}$, approximately a semitone. The difference between these two values is approximately $2.020 \cdot \mathsf{m}_2$.

[33] As a volunteer firefighter, I urge you not to stay in the way of a firetruck. We simply use this model since this makes computations a bit easier. You may use the model of (2.5) for a more realistic approach as a trivial exercise.

2.5. Some examples of interference patterns

In section 2.1 we already discussed a principle called *superposition*. Sometimes this causes some more or less surprising effects.

Lemma 13. *Let $A, B\colon \mathbb{R} \to \mathbb{R}$ be two periodic functions with periods $\alpha > 0$ and $\beta > 0$, respectively. Then the function $S = A + B$ is periodic iff ${}^{\alpha}/_{\beta} \in \mathbb{Q}$.*

Proof. For any $a, b \in \mathbb{Z} \setminus \{0\}$, the functions A and B also have the periods $a\alpha$ and $b\beta$, respectively:

$$\begin{aligned} A(t) &= A(t+\alpha) = A(t+a\alpha), \\ B(t) &= B(t+\beta) = B(t+b\beta) \end{aligned}$$

hold for every $t \in \mathbb{R}$. The function S is periodic iff for some $a, b \in \mathbb{Z} \setminus \{0\}$ the equation $a\alpha = b\beta$ holds, which is equivalent to ${}^{\alpha}/_{\beta} = {}^{b}/_{a} \in \mathbb{Q} \setminus \{0\}$. $\square$

Lemma 14. *For any $A_1, A_2, \omega, \phi_1, \phi_2 \in \mathbb{R}$ there exist real numbers $A, \phi \in \mathbb{R}$, such that*

$$\begin{aligned} A_1 \sin(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_2) &= A \sin(\omega t + \phi), \\ \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1)} &= A, \\ \arg\Big((A_1\cos\phi_1 + A_2\cos\phi_2) + i(A_1\sin\phi_1 + A_2\sin\phi_2)\Big) &= \phi \end{aligned}$$

for all $t \in \mathbb{R}$, in particular

$$\Big||A_1| - |A_2|\Big| \leq A \leq |A_1| + |A_2|. \tag{2.6}$$

Proof. Interpreting the terms of sines as imaginary parts of three complex numbers, the existence is obvious. Using the usual Hilbert space identification of $\mathbb{E}^2$ and $\mathbb{C}^1$, the proof is simple geometry, see [Bro05, section 2.7.3.2] for a sketch. The inequality (2.6) is trivial. $\square$

Corollary 15. *For any $A_1, A_2, \omega, \in \mathbb{R}$ and for all $t \in \mathbb{R}$:*

$$A_1 \sin(\omega t) + A_2 \cos(\omega t) = \sqrt{A_1^2 + A_2^2}\, \sin\Big(\omega t + \arg(A_2 + iA_1)\Big).$$

Lemma 14 allows identifying sums of oscillations of the same frequency. When dealing with the Hilbert space $L^2\left(S^1, \mathbb{R}\right)$, the left hand side of corollary 15 is handier.

One of the simplest cases of constructive interference is given by the following example.

Example 16 (constructive interference). Let us consider an anechoic chamber, then we cannot hear any sound reflections inside the chamber. Let us imagine two loudspeakers at the points $A = 0$ and $B = (4\pi, 0, 0)$, respectively, inside this room. At certain point of time both loudspeakers begin to emit a sine sound wave with the same frequency $(2\pi)^{-1}$ with the same volume. For simplicity, let us assume that the speed of the wave is exactly 1.

Let us have a look on the point $P = (2\pi, 0, 0)$ which is exactly between A and B. Obviously all three points are on one line and we see the distances $|A - P| = 2\pi$ and $|B - P| = 2\pi$ We assume that the amplitude of both sound waves on P is 1. After 2π units of time have passed, the sound wave on point P is given by

$$t \mapsto \sin\Big(t - |A - P|\Big) + \sin\Big(t - |B - P|\Big) = 2 \sin t,$$

which is the same tone emitted by both speakers but louder.

We often encounter constructive interference when several people inside a room speak at the same time.

The following example on destructive interference is a bit more elaborate.

Example 17 (destructive interference). Let us consider an anechoic chamber, then we cannot hear any sound reflections inside the chamber. Let us imagine two loudspeakers at the points $A = 0$ and $B = (3\pi, 0, 0)$, respectively, inside this room. At certain point of time both loudspeakers begin to emit a sine sound wave with the same frequency $(2\pi)^{-1}$. For simplicity, let us assume that the speed of the wave is exactly 1.

Let us have a look on the point $P = (\pi, 0, 0)$. Obviously all three points are on one line and we see the distances $|A - P| = \pi$ and $|B - P| = 2\pi$. We assume that the amplitude of both sound waves on P is 1, hence by the inverse-square law, the amplitude of the sound wave of loudspeaker B is four times greater.

It takes some time for the sound waves to arrive at point P. After 2π units of time have passed, the sound wave on point P is given by

$$t \mapsto \sin\Big(t - |A - P|\Big) + \sin\Big(t - |B - P|\Big) = -\sin t + \sin t = 0,$$

which means that there is no sound at point P. Similar results in a more general setting may be found with lemma 14.

Lemma 18. *Let $a, b \in \mathbb{R}$, then*

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}.$$

Proof. The result is a direct calculation:

$$\begin{aligned}
\sin \frac{a+b}{2} \cos \frac{a-b}{2} &= \Im \exp\left(\frac{a+b}{2} i\right) \Re \exp\left(\frac{a-b}{2} i\right) \\
&= \frac{\exp\left(\frac{a+b}{2} i\right) - \overline{\exp\left(\frac{a+b}{2} i\right)}}{2i} \, \frac{\exp\left(\frac{a-b}{2} i\right) + \overline{\exp\left(\frac{a-b}{2} i\right)}}{2} \\
&= \frac{\exp(ai) - \overline{\exp(ai)} + \exp(bi) - \overline{\exp(bi)}}{4i} \\
&= \frac{\Im \exp(ai) + \Im \exp(bi)}{2} \\
&= \frac{\sin a + \sin b}{2}.
\end{aligned}$$

□

Example 19 (acoustical beat). Let us consider two tones with frequencies $f_1 > 0$ or $f_2 > 0$, respectively sound simultaneously with the same amplitude. Let $f_\mathsf{d} = |f_1 - f_2|$. Then the common sound wave is given by the superposition

$$g(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) = 2 \cos\left(2\pi \frac{f_\mathsf{d}}{2} t\right) \sin\left(2\pi \frac{f_1 + f_2}{2} t\right),$$

compare lemma 18. On the first term the interference becomes clear: If f_d is a very small positive number then the sine factor gives an oscillation similar to that of f_1 or f_2, but the cosine factor is a very slow oscillation, resulting in periodic volume oscillation. This interference pattern is called *acoustical beat* and illustrated in figure 10. In this case the functions $t \mapsto \pm 2\cos(\pi f_\mathsf{d} t)$ are called *envelope functions*.

Identical amplitudes are an obvious restriction to the real world, but this model nonetheless illustrates one basic aspect of this interference patterns quite clearly. Beats may be heard on sustained chords.

The German teacher Walter Fendt illustrates these effects on his web page [Fen16].

2.6. A more detailed view on a handbell's pitch

In this section we want to understand the basics. For further reading and more background, the books [Ros84], [FR98, chapter 21], [FR04], [Ros07], and the video [Ros16] are recommended. As mentioned above the fundamental tone is the most important partial. To understand the so-called *bell formula* which is introduced in section 2.6.1 some background on the vibration of a bell is necessary:

> "The sound spectrum of handbells is quite different from that of church bells. There are usually only two tuned partials, the strike note being determined by the lower one. Handbells are much lighter than church bells of the same pitch and they have no sound bow," [Ros84, page 372]. "The pitch of each bell is determined by the fundamental, supported by one or more harmonic partials. These plus many inharmonic partials determine the timbre of handbell sound. The dominance of the first two harmonic partials of the radiated sound makes it feasible to play several bells together in chords without a clash between the upper partials, as occurs in carillons," [Ros84, pages 390-391].

The casting of a sounding bell oscillates. These movements produce the actual sound by causing oscillations of the surrounding air like any other instruments do. The oscillation is realized through bending. An oscillating bell has several *nodes* which are lines of minimal amplitude. That means simply that the metal is not bending on these lines, see [CMSK99, *retrieved* April 4, 2004]. These lines are also known as *Chladni-patterns*.[34] A tuple of numbers of these lines is called *mode*. Compare figure 11 for an illustration of a few modes:

> "The modes of vibration of bells are described by use of the nomenclature of the normal modes in a circular plate, which they resemble. The $[m, n]$ mode has m meridian nodes extending across the crown (top), and n nodal circles around its periphery. The principal modes are the $[2, 0]$ mode and the $[3, 0]$ mode [… — M. Jedamzik]. Alternate sections of the bell vibrate inward and outward [… — M. Jedamzik]. The strike tone of a handbell is determined by the tuning of the fundamental $[2, 0]$ mode, unlike tuned carillon bells and especially chimes, where the strike tone is a combination tone [… — M. Jedamzik].
>
> The dependence of frequency on mode shape may be compared to that of a cylindrical shell. For a shell with free ends and no circular nodes, the frequencies were found to follow a relationship derived by Rayleigh," [Ros84, pages 390-393].

The formula is introduced in section 2.6.1. Another interesting article on the physics of handbells is given by [CMSK99, *retrieved* April 4, 2004]. The mentioned book in which the formula is said to be derived is [Ray45].

2.6.1. The bell formula

Theorem 20. *The oscillation determined by the mode $[m, 0]$ of a bell with (exterior) diameter d and thickness t, see figure 2, made of a material with Young's modulus E, density ϱ and Poisson's ratio ν oscillates by the*

[34] This pattern was first observed by the German physicist Ernst Florens Friedrich Chladni.

frequency $f_{[m,0]}$ satisfying

$$f_{[m,0]} = \underbrace{\frac{1}{\sqrt{3}\pi}}_{=:T} \cdot \underbrace{\frac{t}{d^2}}_{=:F_{dt}} \cdot \underbrace{\sqrt{\frac{E}{\varrho(1-\nu^2)}}}_{=:B} \cdot \underbrace{\frac{m(m^2-1)}{\sqrt{m^2+1}}}_{=:M_m}, \tag{2.7}$$

see [Ros84, page 393]. The diameter and thickness are measured on the extrema of the vibrating casting corresponding to the node.

For simplicity we write $W_m := TBM_m$, implicating $f_{[m,0]} = W_m F_{dt}$.

Remark 21. (a) According to [Ros84, page 266] this formula is an approximation. This is likely based on the fact that the formula is originally modeled for cylindrical shells which can be compared to *tubular bells* which are used in orchestras. Being an approximation the formula likely does not fit to a given bell perfectly; The formula is nevertheless a valuable tool to understand influences of shape and material to the tone of a bell. We will transfer this question to a Laplacian-spectrum-problem on Riemannian manifolds in section 2.7.

(b) This formula (2.7) is also known as *bell formula*, see [Ros84, page 267] since it allows deriving some important properties of bells.

(c) We note that the formula is independent from the height or weight or volume respectively of the casting.

(d) We note that aspects like sound decay and loudness are two other important acoustical properties which are not covered by the bell formula.

(e) The factor M_m depends on the mode, the factor B on the bell's material, the factor F_{dt} on the shape of the bell and the independent factor T is determined by all mathematical constants.

(f) The constants E and ν are *moduli of elasticity*, which means they measure (in a certain way) the deformation of the material when force is applied to the material. The constant B depends on the material, and small changes likely have a noticeable influence leading to errors.

2.6.2. Simple properties derived by the bell formula

Assuming that the bell is made of non-auxetic material which ensures $\nu > 0$ and that the right hand side of formula (2.7) is an invertible real number, the formula can be solved if all but one variable are given.

Observation 22 (One type of mold for two bells). Since manufacturers in general use one mold for two consecutive bells, we see from (2.7) that the casting of the lower of the two bells has a thickness of about $2^{-1/12} \approx 94.39\%$ compared to the higher pitched bell. This also results in a lighter bell and shows that the principle "lower bells are heavier" is not true in general.

Observation 23 (Equivalence classes of pitch). Formula (2.7) shows that doubling the diameter and multiplying the thickness of the casting with factor 4 gives the same frequencies for all modes $[m, 0]$. This is an important reason why handbells are possible to manufacture: Large church and carillon bells can be scaled down to manufacture handbells of the same fundamental pitch. It also shows that the frequency $f_{[m,0]}$ is proportional to F_{dt}.

Observation 24 (Scaling a bell). The formula (2.7) shows that scaling a bell by factor $\gamma > 0$ scales the frequencies by factor $1/\gamma$, since

$$\frac{1}{\gamma} F_{dt} = F_{\gamma d, \gamma t}. \tag{2.8}$$

Thus scaling a bell by factor 2 simply decreases the pitch by one octave. More general: Choosing $\gamma = \sqrt[12]{2}^{-k}$ increases the pitch by $k \cdot \mathsf{m}_2$. The Calculations $\sqrt[12]{2}^{-1} \approx 0.9439$, $\sqrt[12]{2}^{-(-1)} \approx 1.0595$ show: Scaling the bell by approximately 6% decreases the bell by one semitone, and scaling the bell by approximately −6% increases the bell by one semitone.

Observation 25 (First overtone). Since $M_2 \cdot 2^{18/12} = M_3$ the difference of the two principal modes is exactly $18 \cdot \mathsf{m}_2 = \Big(1 + (12 - 1) + (7 - 1)\Big) \cdot \mathsf{m}_2$ which is a perfect twelfth (one octave and a perfect fifth), as long as the quotients F_{dt} of the corresponding spots coincide.

Observation 26 (Detuning a handbell by grinding). Since handbells are tuned commonly by removing metal from the inside, the parameter t is decreasing while d is invariant, (2.7) yields that handbells are tuned by lowering the frequency. Larger values for the diameter d make tuning easier because removing the same amount of metal has a smaller effect.

Since removing metal from the outside decreases d as well as t, standard arguments show that the pitch of a bell cannot be increased by grinding the outside of the bell iff $4t < d$, since the formula for increasing the pitch by $k \cdot \mathsf{m}_2$ is given by

$$\begin{aligned} & \sqrt[12]{2}^k F_{dt} = F_{d-2\varepsilon,t-\varepsilon} =: F_{dt\varepsilon} && (2.9) \\ \Leftrightarrow \quad & k = \operatorname{ls}(t-\varepsilon) - \operatorname{ls} t + 2\operatorname{ls} d - 2\operatorname{ls}(d-2\varepsilon), \end{aligned}$$

and $\varepsilon \mapsto F_{dt\varepsilon}$ is strictly decreasing for conditions $0 < \varepsilon < t < 4t < d$ (one can alternatively show that k must be non-positive):

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F_{dt\varepsilon} = \frac{4t - d - 2\varepsilon}{(d-2\varepsilon)^3} < 0.$$

Thus the only way to increase the pitch of a bell is decreasing the diameter (as long as the thickness is not decreasing too fast) by grinding the mouth of the bell.

For $\gamma > 0$, we observe

$$F_{\gamma d,\gamma t,\varepsilon} = \frac{1}{\gamma} F_{dt\frac{\varepsilon}{\gamma}} \tag{2.10}$$

which is a generalization of (2.8).

Observation 27 (Modes $[m, 0]$ for large m). Standard arguments show

$$\lim_{m\to\infty} \frac{M_m}{m^2} = 1,$$

hence the asymptote of $m \mapsto f_{[m,0]}$ is $m \mapsto T F_{dt} B m^2$, hence for large m the map $f_{[\,\cdot\,,0]}$ behaves like a parabola and the frequencies of the modes $[m, 0]$ grow rapidly and do not vanish.

Observation 28 (Tones on different places of the casting). The values t, d can be seen as functions $t, d \in C^2\Big((0, H), \mathbb{R}_{>0}\Big)$ dependent of the position h.[35] The simple calculation with

$$F_{dt}(h) = \frac{t}{d^2}(h),$$
$$\frac{\dot{t}}{t} \leq 2\frac{\dot{d}}{d} \tag{2.11}$$

shows

$$\dot{F}_{dt}(h) = \frac{\mathrm{d}}{\mathrm{d}h}\frac{t}{d^2}(h) = \frac{\dot{t}d^2 - 2td\dot{d}}{d^4}(h) = \frac{\dot{t}d - 2t\dot{d}}{d^3}(h) \leq 0,$$

which just means the frequency (of the modes emitted on the maximum amplitude) is higher on the vibrating spots near the neck than on those spots near the lip. Note that the bell formula likely is more inaccurate for small h. The inequality (2.11) is satisfied with $\dot{t} \leq 0 \leq \dot{d}$, which just means that the diameter of the casting increases from the shoulder to the lip, while the thickness decreases.

One note to the geometry of a bell: Section 2.7 shows that the requirements for this statement can be satisfied.

Observation 29 (Natural classes of bell sets). For any positive real numbers $\alpha, f > 0$ and any $\epsilon \in \mathbb{R}$ and a fixed mode $[m, 0]$ choosing the diameter d and thickness t_m by

$$d = \frac{\alpha}{f^\epsilon} \cdot \mathsf{m}, \qquad t_m = \frac{\alpha^2}{f^{2\epsilon-1}W_m} \cdot \frac{\mathsf{m}^2}{\mathsf{s}}, \qquad \alpha < \frac{1}{2}W_m f^{\epsilon-1} \cdot \frac{\mathsf{s}}{\mathsf{m}} \tag{2.12}$$

determines a bell, which suffices

$$f_{[m,0]} = W_m F_{dt_m} = f \cdot \mathsf{Hz}.$$

Obviously the definition

$$\widetilde{t} = \frac{\widetilde{\alpha}}{f^{\widetilde{\epsilon}}} \cdot \mathsf{m}, \qquad \widetilde{d}_m = \sqrt{\frac{\widetilde{\alpha}W_m}{f^{1+\widetilde{\epsilon}}} \cdot \mathsf{s} \cdot \mathsf{m}}, \qquad \widetilde{\alpha} < \frac{1}{4}W_m f^{\widetilde{\epsilon}-1} \cdot \frac{\mathsf{s}}{\mathsf{m}} \tag{2.13}$$

[35] It is reasonable to assume our model does not start at the neck of the bell, but rather on the shoulder/waist section.

giving a thickness-modeling priority is equivalent. The necessary inequalities of (2.12) and (2.13) are natural but likely too weak for designing real bells where slightly different values on both sides are to be avoided. Without loss of generality we consider the definition (2.12) instead of (2.13).

Note that with this definition (2.12) only the frequency of one mode $[m, 0]$ can be chosen freely. The natural mode is the fundamental: $m = 2$.

Note that for $\epsilon = 0$ the diameter is independent from f and the thickness is independent from f for $\epsilon = 1/2$. This implies that the diameter or the thickness, respectively can be chosen arbitrarily.

For $\epsilon = 1$ the values t_m and d are both proportional to $1/f$. This is the only configuration for ϵ where both variables are proportional to the same exponentiation $f^{-\epsilon}$.

A bell foundry can choose a fixed α and a fixed ϵ to cast a set of congruent bells. Obviously the inequalities of (2.12) and (2.13) need to be satisfied for all f of the bell set; Since the right hand sides of the inequalities of (2.12) and (2.13) are basically power functions $(f \mapsto f^r)\Big|_{\mathbb{R}_{>0}}$ testing the lowest and the highest frequency of the set is sufficient.

According to [Ros84, pages 402-404] these measurements with $\epsilon = 1$ were used widely on carillons in the 15^{th} and 16^{th} centuries and these measurements with $\epsilon = 1/2$ are used on manufacturer-depending subsets of handbells.

2.6.3. Examples for different materials

The common alloy classified in the *unified numbering system* (UNS) with number *C91300* (approximately 81% copper, 19% tin) is expected to have a similar composition as the bronze mentioned in section 1.7.1.

Example 30 (Bronze). To calculate an example, we consider the values $E = 117 \cdot 10^9 \cdot \mathsf{Pa}$ for Young's modulus, $\nu = 0.34$ for Poisson's ratio and density $\varrho = 8640 \cdot \mathsf{kg} \cdot \mathsf{m}^{-3}$, see [AZo00, *Copper Casting Alloy UNS C91300, retrieved* August 22, 2013]. Noting $\mathsf{Pa} = \mathsf{N} \cdot \mathsf{m}^{-2}, \mathsf{N} = \mathsf{kg} \cdot \mathsf{m} \cdot \mathsf{s}^{-2}$ and

$\mathsf{Hz} = \mathsf{s}^{-1}$ allows computing easily

$$\begin{aligned} f_{[2,0]} &= \frac{25000}{2211\pi}\sqrt{287430}\frac{t \cdot \mathsf{m}}{d^2} \cdot \mathsf{Hz} \\ &= \frac{125 \cdot 10^6}{280797\pi}\sqrt{287430}\frac{t \cdot \mathsf{inch}}{d^2} \cdot \mathsf{Hz} \\ &\approx 75968.53024\frac{t \cdot \mathsf{inch}}{d^2} \cdot \mathsf{Hz}. \end{aligned} \tag{2.14}$$

The parameters t and d are located on the mouth of the bell, see figure 2. Working with $t = \frac{21}{128} \cdot \mathsf{inch} = 0.41671875 \cdot \mathsf{cm}$ and $d = \left(10 + \frac{165}{256}\right) \cdot \mathsf{inch} = 27.037109375 \cdot \mathsf{cm}$ yields $f_{[2,0]} \approx 109.9993 \cdot \mathsf{Hz}$, which is approximately the fundamental tone of the A_2, see table 6. In fact the A_2 is approximately $0.0108 \cdot \mathsf{ct}$ higher. These values for t, d imply $\alpha \approx 2.835667$ for $\epsilon = 1/2$ as in the model of (2.12).

Example 31 (Aluminium). We now compare bronze with pure aluminium. We consider the values $\widehat{E} = 863 \cdot 10^8 \cdot \mathsf{Pa}$ for Young's modulus, $\widehat{\nu} = 0.34$ for Poisson's ratio and density $\widehat{\varrho} = 2689.8 \cdot \mathsf{kg} \cdot \mathsf{m}^{-3}$, see [AZo00, *Aluminium – Specifications, Properties, Classifications and Classes, Supplier Data by Aalco, retrieved* August 23, 2013]. Again a simple calculation for $\widehat{W_2}$ for the pure aluminium gives

$$\begin{aligned} \widehat{W_2} &= \frac{1000000}{9911913\pi}\sqrt{8553980919} \cdot \mathsf{m} \cdot \mathsf{Hz} \\ &\approx 116934.60900 \cdot \mathsf{inch} \cdot \mathsf{Hz}, \\ \frac{\widehat{W_2}}{W_2} &= \frac{4}{58279}\sqrt{502947770} \qquad (2.15) \\ &\approx 1.53925 \\ &\approx 1.24067^2 \qquad (2.16) \\ &\approx 0.64967^{-1}, \qquad (2.17) \end{aligned}$$

which just means that the frequencies of an aluminium bell compared to a bronze bell of the same shape are

$$\left(\operatorname{ls}\widehat{W_2} - \operatorname{ls} W_2\right) \cdot \mathsf{m}_2 \approx 7.47 \cdot \mathsf{m}_2$$

higher. From (2.16) we deduce that a bronze handbell has the same frequencies as an aluminium bell with a diameter of approximately 124% and the same thickness. From (2.17) we deduce that a bronze handbell has the same frequencies as an aluminium bell with a thickness of approximately 65% and the same diameter. In the A_2 example this would be $\widehat{t} \approx \frac{27}{256} \cdot \mathsf{inch} \approx 0.2707 \cdot \mathsf{cm}$ or $\widehat{d} \approx \left(13 + \frac{53}{256}\right) \cdot \mathsf{inch} \approx 33.5440 \cdot \mathsf{cm}$ respectively.

Example 32 (Brass). The values $\widetilde{E}, \widetilde{\nu}$ and $\widetilde{\varrho}$ of the brass alloy [AZo00, *Brass Alloy UNS C44300*, *retrieved* August 27, 2013] (approximately 71% copper, 28% zinc, 1% tin; a similar alloy is used on some brass instruments) differs from C91300 only in terms of density: $\widetilde{\varrho} = 8530 \cdot \mathsf{kg} \cdot \mathsf{m}^{-3}$. This implies

$$\frac{\widetilde{W_2}}{W_2} = \sqrt{\frac{\widetilde{\varrho}^{-1}}{\varrho^{-1}}} \approx 1.00643 \approx \sqrt[12]{2}^{0.111},$$

which just means that the frequencies of a brass bell compared to a bronze bell of the same shape are approximately $0.111 \cdot \mathsf{m}_2$ higher.

Remark 33. Similar calculations can be made to any solid material: While cork is perhaps a very uncommon material, the metals gold and especially lead (plumbum) lead to very small bells compared to bronze, in other words theses metals have the opposite effect of aluminium with greater values.

Another interesting class of materials might be nanometal-polymer hybrids, probably more theoretically than practical since tuning might be a lot more difficult.

Finally a simple model for comparison of weights:

Observation 34. Let $W_m, B, E, \rho, \nu, d, t$ be characteristic data of bell 1 and let $\widehat{W_m}, \widehat{B}, \widehat{E}, \widehat{\rho}, \widehat{\nu}, \widehat{d}$ be data of bell 2 in the notion of theorem 20. As seen above, different values might imply a different pitch of the two bells. However, by scaling a bell we might get two bells of different size and materials but with the same pitch. A natural point of interest is the scale factor and the ratio of the weights of the two bells.

First we just use a trivial way to scale the bell: Every dimension of bell 2 is the dimension of the bell 1 scaled with factor $\gamma > 0$, for example $\widehat{d} = \gamma d$. This implies

$$F_{dt} W_m = f_{[m,0]} = \widehat{f}_{[m,0]} = \frac{1}{\gamma} F_{dt} \widehat{W}_m \tag{2.18}$$

yielding

$$\gamma = \frac{\widehat{W}_m}{W_m} = \frac{\widehat{B}}{B}. \tag{2.19}$$

Note that γ is independent from m, and thus both bells share the same tones of mode $[m, 0]$ for all m.

Let V be the volume of bell 1 and M its mass, similarly let $\widehat{V}$ be the volume of bell 2 and $\widehat{M}$ its mass, thus

$$\widehat{V} = \gamma^3 V,$$

hence

$$\begin{aligned}
\widehat{M} = \widehat{V}\widehat{\rho} = \gamma^3 V \widehat{\rho} &= \gamma^3 \frac{M}{\rho} \widehat{\rho} \\
&= \frac{\widehat{\rho}}{\rho} \left(\frac{\widehat{B}}{B} \right)^3 M \\
&= \sqrt{\frac{\rho}{\widehat{\rho}} \left(\frac{(1-\nu^2)\widehat{E}}{(1-\widehat{\nu}^2)E} \right)^3} M.
\end{aligned} \tag{2.20}$$

Similarly, if we do not scale the height, but only the other two dimensions, we get:

$$\begin{aligned}
\widehat{M} &= \frac{\widehat{\rho}}{\rho} \left(\frac{\widehat{B}}{B} \right)^2 M \\
&= \frac{(1-\nu^2)\widehat{E}}{(1-\widehat{\nu}^2)E} M,
\end{aligned} \tag{2.21}$$

thus the factor is independent from the densities as long as Poisson's ratio and Young's modulus are. This second model is valid, since the only geometric parameters of the bell formula are the diameter and the thickness and (2.19) is independent from geometric parameters.

Example 35. Comparing bronze and aluminium with the models of observation 34, we compute with (2.15) and (2.20):

$$\begin{aligned}\widehat{M} &= \frac{1726}{34093215}\sqrt{502947770}M \\ &\approx 1.13536M,\end{aligned}$$

hence the aluminium casting would be approximately 13.5% heavier.

If we use (2.21) instead of (2.20), we get

$$\begin{aligned}\widehat{M} &= \frac{863}{1170}M \\ &\approx 0.73761M,\end{aligned}$$

hence the aluminium casting would be approximately 26.2% lighter.

Note that this model only focuses on the weight of the casting.

Remark 36. We neglected the *warble effect* caused by varying parameters which occurs often considering that bells are neither really radial symmetric nor of really homogeneous alloy. It is not even clear that the factor B varies continuously on the point of the casting. The real frequency of the $[2,0]$ mode is then given by the positive difference of two corresponding node frequencies, see [Ros84, page 391] for more details.

Remark 37 (atomic radius and polishing). The molar mass M_{B} of one average atom and the density ϱ_{B} of the bronze alloy [AZo00, *Copper Casting Alloy UNS C91300*, *retrieved* August 22, 2013] are

$$\begin{aligned}M_{\mathsf{B}} &\approx \Big(0.81 \cdot 63.540 + (1-0.81)\cdot 118.71\Big)\cdot \mathsf{g}\cdot\mathsf{mol}^{-1} = 74.0223\cdot\mathsf{g}\cdot\mathsf{mol}^{-1}, \\ \varrho_{\mathsf{B}} &= 8640000\cdot\mathsf{g}\cdot\mathsf{m}^{-3}.\end{aligned}$$

The AVOGADRO constant[36] is $N_{\mathrm{A}} \approx 6.022140857 \cdot 10^{23} \cdot \mathsf{mol}^{-1}$, compare [Nat14, *CODATA Value: Avogadro constant*, *retrieved* March 08, 2017]. Assuming a simple cubic lattice system, a simple calculation gives the *metallic radius*

$$r_{\mathrm{B}} = \frac{1}{2}\sqrt[3]{\frac{{}^{6}\!/\!{}_{\pi} M_{\mathsf{B}}}{\varrho_{\mathsf{B}} N_{\mathrm{A}}}} \approx 150.315271 \cdot 10^{-12} \cdot \mathsf{m} \tag{2.22}$$

of an average atom in the bronze alloy, the factor

$$\frac{d_{\mathrm{B}}^{3}}{\frac{8}{8} \cdot \frac{4}{3}\pi \left(\frac{d_{\mathrm{B}}}{2}\right)^{3}} = \frac{6}{\pi}$$

adjusts the volume of the sphere packing. The *metallic diameter* thus is $d_{\mathrm{B}} = 2r_{\mathrm{B}}$.

Some people believe that polishing does change the tone of a handbell by removing small layers of atoms. Using the values $(k, \epsilon) = (-{}^{1}\!/\!{}_{100}, x d_{\mathrm{B}})$ and t, d as in example 30 in equation (2.9), we see by solving this equation for x, that we need to remove approximately 8530 layers of atoms which corresponds to approximately $0.0025644 \cdot \mathsf{mm}$, on the outside of the bell to decrease the tone of the bell by $1 \cdot \mathsf{ct}$ and approximately 42594 layers of atoms which corresponds to approximately $0.0128051 \cdot \mathsf{mm}$, on the outside of the bell to decrease the tone of the bell by $5 \cdot \mathsf{ct}$.

We now consider a bell of the same thickness but of diameter ${}^{d}\!/\!{}_{2}$, hence the new bell's fundamental tone is two octaves higher. Removing the same amount of layers results in lowering the tone by approximately $0.00934 \cdot \mathsf{ct}$ or $0.04672 \cdot \mathsf{ct}$, respectively. The differences compared with the larger bell are approximately $0.00066 \cdot \mathsf{ct}$ or $0.00328 \cdot \mathsf{ct}$ respectively.

2.6.4. Final remark

As noted in the beginning of this chapter, the discussed model is a simplification. Nevertheless this model based on only one formula allows for deriving some interesting properties.

[36]This constant is named after the Italian Scientist LORENZO ROMANO AMEDEO CARLO AVOGADRO.

The unavoidably-inhomogeneous alloy makes a precise calculation of a bell's tone practically impossible. The mold of a handbell is very thick compared to a ground casting and is ground successively (in contrast to church bells in general) by experienced artisans to get the desired sound to meet the needs of handbell music which includes the ability to ring chords with several bells in proper harmonics. Having this in mind, the presented formula is an appropriate approximation on the real needs although it is only one aspect of bell founding.

Fast computer systems and research allow complex models of a bell to simulate its tone even more precisely. One of the used tools is the well-known finite element method. Bell foundries like *Koninklijke Eijsbouts* claim the use of a very precise model for computer-based simulations of a bell, which allows for precise predictions on designed bells, compare [Kon17, *Products & Services. Bells/Church Bells/Stationary Bells*, *retrieved* March 6, 2017].

Example 38 (frequency of an organ pipe). One might want to compute the length of organ pipes for comparison. There are actually two types of pipes: *closed end* (or *gedackt*) and *open end*. A closed end pipe with length L_{g} has the fundamental frequency f while the speed of sound is denoted by c, satisfying:

$$L_{\mathsf{g}} = \frac{c}{4f}.$$

The situation on an open pipe is more complicated, see [Ray45, section 255, formula (8) and section 322a]. Open-ended pipes are approximately twice as long.

Applying this to a C_2, we get $L_{\mathsf{g}} \approx 1.3118 \cdot \mathsf{m} \approx \left(51 + \frac{41}{64}\right) \cdot \mathsf{inch}$.

2.7. On the geometry of a bell

This section is (almost) pure mathematics. For the first time we use some nontrivial mathematics in this book: In this section we discuss the casting of the handbell as a smooth Riemannian manifold[37] without boundary. For convenience we identify tensor fields of rank one or two with matrices.

The exterior boundary of the casting can be parameterized, when considering a cross section of the bell, with the graph

$$\Gamma_r = \Big(h \mapsto \big(h, r(h)\big)\Big) : (0, H) \to (0, H) \times \mathbb{R}_{>0}\,. \tag{2.23}$$

of $r := d/2$. From now on we do not necessarily assume that Γ_r is parameterized as a graph as in (2.23), but is more likely a parameterized regular curve

$$\Gamma_r \colon (0, H) \to \mathbb{R}^2_{>0}$$

with $\dot{\Gamma}^1_r \geq 0$. Let

$$t \colon (0, H) \to \mathbb{R}_{>0}$$

be a bounded continuously differentiable function, which models the thickness of the casting: On $\Gamma_r(h)$ the thickness of the casting is exactly $t(h)$. The interior boundary can be parameterized via the offset curve

$$\Pi_{rt} := \Gamma_r - \frac{t}{\left|\dot{\Gamma}_r\right|} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{=:J} \dot{\Gamma}_r,$$

yielding the necessary condition

$$0 < \Pi^2_{rt} = \Gamma^2_r - \frac{t}{\left|\dot{\Gamma}_r\right|} \dot{\Gamma}^1_r \tag{2.24}$$

or in case of the graph of (2.23):

$$t < r\sqrt{1 + \dot{r}^2}, \tag{2.25}$$

[37] *Riemannian Geometry* is an important branch of *differential geometry* and is named after Georg Friedrich Bernhard Riemann. For some mathematical background, see [Jos11], [KN96a, KN96b], or [Spi99a, Spi99b, Spi99c, Spi99d, Spi99e].

which is obviously sharper than $t < r$.

The geometrically canonical condition for the image of Π_{rt} is to be a subset of $\mathbb{R}^2_{>0}$. A restriction to a sheet of the covering map

$$\begin{aligned}\Sigma_{rt}\colon (0,H) \times (0,1) \times \mathbb{R} &\to \mathbb{R}^3 \\ (h,s,\phi) &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & \cos\phi \\ 0 & \sin\phi \end{pmatrix} \Pi_{r,st}(h)\end{aligned}$$

of the casting gives a parameterization of the casting (sliced, without the axis of symmetry[38] and boundary) as long as t, d are well-behaved:

Lemma 39. *Let the image of the curve Π_{rt} be in first quadrant of the plane and d a convex function. Then the map Σ_{rt} is an embedding.*

Proof. The usual arguments with a to arc-length parameterized curve Γ_r and a calculation of the Jacobian determinant of Σ_{rt} yield

$$\det\left(\mathrm{D}\Sigma_{rt}\right)(\,\cdot\,,s,\phi) = \left(st\ddot{r}\frac{\dot{r}^2+\dot{r}+1}{\sqrt{1+\dot{r}^2}^3}+1\right)t\Pi^2_{r,st}. \tag{2.26}$$

The second factor t is positive by definition. If the function d is convex, then the first factor is positive. Since the last factor is exactly the second coordinate of $\Pi_{r,st}$ it is positive, too. These conditions ensure Σ_{rt} to be an immersion. The conditions $\dot{d}, \ddot{d} \geq 0$ ensure that Σ_{rt} is a homeomorphism from its preimage to its image and thus an embedding: The equation $\phi = \arg(\Sigma^3_{rt}, \Sigma^2_{rt})(h,s,\phi)$ is self-evident and h can be reconstructed by the orthogonal projection on the trace of the curve Γ_r which then leads to the value s. □

Observation 40. The submanifold Σ_{rt} can be equipped with the Riemannian pullback-metric

$$g\big|_{(h,s,\phi)} = \left(\Sigma^*_{rt}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{E}^3}\right)\Big|_{(h,s,\phi)} = \left(\mathrm{D}\Sigma^T_{rt}\cdot\mathrm{D}\Sigma_{rt}\right)\Big|_{(h,s,\phi)},$$

[38] The castings of handbells made by Malmark and Schulmerich are actually homeomorphic to $S^1 \times [0,1]^2$, not $[0,1]^3$.

induced by the scalar product of $\mathbb{E}^3$, giving the bell the structure of a Riemannian manifold. Although the calculation is easy, the resulting coefficients are cumbersome: Since

$$D\Sigma_{rt}(\,\cdot\,,s,\phi) = \left(\begin{pmatrix} 1 & 0 \\ 0 & \cos\phi \\ 0 & \sin\phi \end{pmatrix}\begin{pmatrix} \dot{\Pi}_{r,st} & \frac{\partial}{\partial s}\Pi_{r,st} \end{pmatrix} \quad \begin{pmatrix} 0 \\ -\sin\phi \\ \cos\phi \end{pmatrix}\Pi^2_{r,st}\right),$$

we calculate

$$\begin{aligned} g(\,\cdot\,,s,\phi) &= \begin{pmatrix} \left|\dot{\Pi}_{r,st}\right|^2 & \left\langle \dot{\Pi}_{r,st}, \frac{\partial}{\partial s}\Pi_{r,st} \right\rangle & 0 \\ \left\langle \frac{\partial}{\partial s}\Pi_{r,st}, \dot{\Pi}_{r,st} \right\rangle & \left|\frac{\partial}{\partial s}\Pi_{r,st}\right|^2 & 0 \\ 0 & 0 & \left(\Pi^2_{r,st}\right)^2 \end{pmatrix} \\ &= \begin{pmatrix} \left|\dot{\Pi}_{r,st}\right|^2 & st\dot{t} & 0 \\ st\dot{t} & t^2 & 0 \\ 0 & 0 & \left(\Pi^2_{r,st}\right)^2 \end{pmatrix}, \end{aligned} \tag{2.27}$$

with

$$\begin{aligned} G &:= \frac{1}{\left|\dot{\Gamma}_r\right|} \cdot \dot{\Gamma}_r, \\ \kappa_{\Gamma_r} &= \frac{\left\langle J\dot{\Gamma}_r, \ddot{\Gamma}_r \right\rangle}{\left|\dot{\Gamma}_r\right|^3}, \\ \Pi^2_{r,st} &= \Gamma^2_r - stG^1, \end{aligned}$$

$$\begin{aligned}
\left|\dot{\Pi}_{r,st}\right|^2 &= \left|\dot{\Gamma}_r - s\dot{t}JG - stJ\dot{G}\right|^2 \\
&= \left|\dot{\Gamma}_r\right|^2 + \left(s\dot{t}\right)^2 |G|^2 + (st)^2 \left|\dot{G}\right|^2 \\
&\quad - 2s\dot{t}\left\langle \dot{\Gamma}_r, JG \right\rangle - 2s^2t\dot{t}\left\langle G, \dot{G}\right\rangle - 2st\left|\dot{\Gamma}_r\right|\left\langle G, J\dot{G}\right\rangle \\
&= \left|\dot{\Gamma}_r\right|^2 + \left(s\dot{t}\right)^2 + (st)^2\left|\dot{G}\right|^2 + 2st\left|\dot{\Gamma}_r\right|\left\langle JG, \dot{G}\right\rangle \\
&= \left|\dot{\Gamma}_r\right|^2 + \left(s\dot{t}\right)^2 + (st)^2\left|\dot{G}\right|^2 - 2st\left|\dot{\Gamma}_r\right|^2 \kappa_{\Gamma_r}.
\end{aligned} \tag{2.28}$$

Since its determinant is simply $\det^2 (\mathrm{D}\Sigma_{rt})(\,\cdot\,, s, \phi)$ as in (2.26) the tensor is obviously symmetric and non-degenerate: Lemma 39 shows, that the eigenvalues of g are positive.

Example 41. For any $\theta \in \Big(-{}^{\pi}/_{2}, {}^{\pi}/_{2}\Big)$ and any $r_0 \geq 0$, satisfying (2.24), let

$$\begin{aligned}
\Gamma_r \colon (0, H) &\to \mathbb{R}^2_{>0} \\
h &\mapsto \begin{pmatrix} h\cos\theta \\ r_0 + h\sin\theta \end{pmatrix},
\end{aligned}$$

thus Γ_r is line parameterized to arc-length. From (2.27) we calculate the Riemannian metric

$$g(h, s, \phi) = \begin{pmatrix} 1 + \left(s\dot{t}(h)\right)^2 & s(t\dot{t})(h) & 0 \\ s(t\dot{t})(h) & t^2(h) & 0 \\ 0 & 0 & \varpi^2(h, s) \end{pmatrix}, \tag{2.29}$$

where $\varpi(h, s) = \Pi^2_{r,st}(h) = r_0 + h\sin\theta - st(h)\cos\theta$.

We first make some trivial calculations[39] for a function, which we will need later for a more subtle argument:

[39] This observation can easily generalized to a strictly monotonic, function instead of $\sqrt{\cdot}$, with some obvious restrictions.

Observation 42. Let $f \in C^2(0,H)$ be a function with $0 < f(h)$ and $f''(h) \neq 0$ for some $h \in (0,H)$. Then $f(h)$ is a local maximum or local minimum, respectively, iff $\sqrt{f(h)}$ is a local maximum or local minimum, respectively.

Proof. Simple calculations and standard arguments show the statement with an existing second derivative of $\sqrt{f}$:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\xi}\right|_{\xi=h} \sqrt{f(\xi)} = \frac{f'}{2\sqrt{f}}(h),$$

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}^2\xi}\right|_{\xi=h} \sqrt{f(\xi)} = \frac{2ff'' - f'^2}{4\sqrt{f}^3}(h). \qquad \square$$

Our model of a bell actually realizes the thickness as intended:

Proposition 43. *Let $h \in (0,H)$ be a number and d a convex function. Then the distance between the trace of Γ_r and the point $\Pi_{rt}(h)$ has a (local) minimum at h with value $t(h)$.*

Proof. Without loss of generality Γ_r is parameterized to arc-length, with length H of the curve Γ_r. With $D_{rth} := \xi \mapsto \left|\Gamma_r(\xi) - \Pi_{rt}(h)\right|$ and $D^2_{rth} \in C^2(0,H)$ simple calculations show

$$\begin{aligned} D_{rth}(h) &= |\Gamma_r - \Pi_{rt}|(h) \\ &= t(h) > 0, \end{aligned} \tag{2.30}$$

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\xi}\right|_{\xi=h} D^2_{rth}(\xi) &= 2\left.\left\langle \dot\Gamma_r(\xi), \Gamma_r(\xi) - \Pi_{rt}(h)\right\rangle\right|_{\xi=h} \\ &= 2t(h)\left\langle \dot\Gamma_r, J\dot\Gamma_r\right\rangle(h) \\ &= 0, \end{aligned} \tag{2.31}$$

$$\begin{aligned} \left.\frac{\mathrm{d}^2}{\mathrm{d}^2\xi}\right|_{\xi=h} D^2_{rth}(\xi) &= 2\left.\left(\left\langle \ddot\Gamma_r(\xi), \Gamma_r(\xi) - \Pi_{rt}(h)\right\rangle + \left\langle \dot\Gamma_r, \dot\Gamma_r\right\rangle(\xi)\right)\right|_{\xi=h} \\ &= 2t(h)\left\langle \ddot\Gamma_r, J\dot\Gamma_r\right\rangle(h) + 2 \\ &= 2t(h)\kappa_{\Gamma_r}(h) + 2 \end{aligned} \tag{2.32}$$

with curvature κ_{Γ_r} of Γ_r. The convexity of $d = 2r$ just means $\ddot{r} \geq 0$ which ensures that (2.32) is positive. Observation 42 and the equations (2.31), (2.32) and (2.30) ensure that the distance between Γ_r and $\Pi_{rt}(h)$ is a (local) minimum at h with value $t(h)$. $\square$

We recall all assumptions for the case we derived: $d = 2r, t \in C^2(0, H)$ with $0 < t < r$, while d is a convex and isotone function $(0 \leq \dot{d}, \ddot{d})$ and t is an antitone function $(\dot{t} \leq 0)$. These assumptions are derived in the formulas (2.11), (2.25), (2.26) and (2.32) and are stronger than necessary, but natural. The convexity of $d = 2r$ is satisfied for usual shapes of bells. Figure 2 is an example for a convex function diameter, except the shoulder and waist part.

We will give a short introduction to the calculation of the overtones.

Definition 44. The *Laplace-Beltrami operator*[40] or *Laplacian* of a smooth function u on the casting is defined by

$$\Delta u = \frac{1}{\sqrt{g}} \partial_i \Big(\sqrt{g} g^{ij} \partial_j u \Big). \tag{2.33}$$

Example 45. Let us look on the Laplacian of observation 40 with constant thickness. Therefore let

$$\begin{aligned} p(h,s) &= \Big| \dot{\Pi}_{r,st}(h) \Big|, \\ q(h,s) &= \Pi^2_{r,st}(h), \\ t &= \text{const.}, \end{aligned}$$

then the Laplacian of a function u is given by

$$\Delta u = \left(\frac{1}{p^2} u_{;hh} + \frac{q_{;h} p - q p_{;h}}{p^3 q} u_{;h} \right) + \frac{1}{t^2} \left(u_{;ss} + \frac{p_{;s} q + p q_{;s}}{pq} u_{;s} \right) + \frac{u_{;\phi\phi}}{q^2}.$$

[40] The differential operator is named after the French mathematician Pierre-Simon, marquis de Laplace and the Italian mathematician Eugenio Beltrami. Note the sign convention.

Example 46. For $t \equiv$ const., we get a simple form of example 41. In this case $\theta = \pi/2$ is not problematic. The metric (2.29) simplifies to

$$g(h, s, \phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^2 & 0 \\ 0 & 0 & \varpi^2(h, s) \end{pmatrix},$$

implying

$$\Delta u = \left(u_{;hh} + \frac{\sin\theta}{\varpi} u_{;h} \right) + \frac{1}{t^2} \left(u_{;ss} - \frac{t\cos\theta}{\varpi} u_{;s} \right) + \frac{1}{\varpi^2} u_{;\phi\phi}. \tag{2.34}$$

Definition 47. An *eigenfunction u of the Laplacian* to the *eigenvalue* $\lambda \in \mathbb{R}$ satisfies

$$\Delta u = \lambda u.$$

The corresponding partial differential equation is called the *Helmholtz equation*. The set of all eigenvalues is the *spectrum* of the Laplacian.

The *wave equation* is the partial differential equation

$$c^2 \Delta u = \frac{\partial^2}{\partial^2 \tau} u, \tag{2.35}$$

where τ is the time and $c \in \mathbb{R}$ is the *propagation speed of the wave*, a constant depending on the material of the casting.

The *d'Alembert operator*, *d'Alembertian* or *wave operator*[41] is the linear second order differential operator

$$\Box_c = c^2 \Delta - \frac{\partial^2}{\partial^2 \tau},$$

hence (2.35) is equivalent[42] to $\Box_c u = 0$ and solutions of the wave equation are exactly elements of the kernel of $\Box_c$.

[41] This operator is named after the French mathematician Jean-Baptiste le Rond d'Alembert.

[42] Note the sign convention here.

Observation 48. For all constants $c \in \mathbb{R}^*$, the wave operators $\Box_c$ are equivalent: Let $c, \widetilde{c} \in \mathbb{R}^*, q = \left|\tilde{c}/c\right|$ and $u\colon \Sigma_{rt} \times \mathbb{R} \to \mathbb{R}$ be a smooth function, and

$$\widetilde{u}\colon (x, \tau) \mapsto \frac{1}{q^2} u(x, q\tau).$$

Then the simple calculation

$$\begin{aligned}(\Box_{\tilde{c}} \tilde{u})\Big|_{(x,\tau)} &= \frac{\tilde{c}^2}{q^2} \Delta\Big(u(x, q\tau)\Big) - \frac{\partial^2}{\partial^2 \tau}\left(\frac{1}{q^2} u(x, q\tau)\right) \\ &= \left(c^2 \Delta u - \frac{\partial^2}{\partial^2 \tau} u\right)\Bigg|_{(x,q\tau)} = (\Box_c u)\Big|_{(x,q\tau)}\end{aligned}$$

shows the transformation. But this identification obviously scales the time-dimension, therefore we do not normalize the wave-operator to $\Box_1$.

Remark 49. A bell is an instrument of the class *idiophone*. Idiophones create the sound by vibrating as a whole without additional strings or membrane. Therefore the wave equation is a plausible tool for modeling the oscillations of a bell.

The wave equation is an important example of a partial differential equation. For general information on the wave equation in the Euclidean space, see [Eva98, section 2.4] for a standard reference.

A string instrument with a sound box for example is much more complicated.

Observation 50. Let $\lambda \in \mathbb{R}$ be a real number and $u\colon \mathbb{R}^3 \to \mathbb{R}$ be a function such that

$$\Delta u = -\lambda^2 u, \qquad \frac{\partial}{\partial \tau} u \equiv 0.$$

Then for any $\tau_0 \in \mathbb{R}$, the function

$$\widehat{u}(h, s, \phi, \tau) := u(h, s, \phi) \cdot \sin(\lambda c \tau + \tau_0) \tag{2.36}$$

is a solution of the wave equation. Since

$$\begin{aligned}\sin(\lambda c\tau + \tau_0) &= \sin(-\lambda c\tau + \pi - \tau_0)\\ &= \cos\Big(\lambda c\tau - \pi/2 + \tau_0\Big)\\ &= \cos\Big(-\lambda c\tau + \pi/2 - \tau_0\Big),\end{aligned}$$

without loss of generality $c\lambda \geq 0$, hence the frequency of oscillations of solution (2.36) is

$$\frac{c\lambda}{2\pi} \cdot \mathsf{Hz}, \tag{2.37}$$

as long as $\widehat{u}$ satisfies the initial value problem and τ models the time in seconds.

The wave equation is linear: Let u, w be solutions of the wave equation. Then for any constant $a \in \mathbb{R}$ the function $(u + aw)$ is also a solution. This corresponds to the fact that the tone decomposes into the fundamental tone and several overtones.

Thus a solution of the Helmholtz equation gives a solution of the wave equation.

Observation 51. The helpful result[43] [Jos11, Theorem 3.2.1] states that the eigenvalue problem on a connected and compact Riemannian manifold has exactly countable many eigenvalues. All of them are nonpositive: We denote them by $\Big(-\lambda_k^2\Big)_{k\in\mathbb{N}_0}$ with corresponding eigenfunctions v_k, hence the Helmholtz equation

$$\Delta v_k = -\lambda_k^2 v_k$$

holds. Without loss of generality, $\lambda_k \geq 0$, compare observation 50. These eigenvalues satisfy

$$\begin{aligned}\lambda_k = 0 &\Leftrightarrow k = 0,\\ \lambda_k^2 &\leq \lambda_{k+1}^2,\\ \lim_{k\to\infty} \lambda_k^2 &= \infty.\end{aligned}$$

[43]Note that our definition of the Laplacian differs by sign.

Therefore, the multiplicity of any eigenvalue is finite. The eigenvectors v_k form a $L^2(\Sigma_{rt})$-orthonormal Schauder[44] basis of solutions inside the Sobolev[45] space $W^{2,2}(\Sigma_{rt})$; hence series of the eigenfunctions multiplied with sine functions are solutions of the wave equation, compare observation 50. This also implies that the degree of freedom is countable infinity. As a consequence, the sound of a bell cannot be calculated without additional information. This is not really surprising as different techniques, different striking points and different materials of the striking objects make different sounds on the same casting.

Observation 27 is compatible to growth of the sequence $\left(\lambda_k^2\right)_{k\in\mathbb{N}}$.

Lemma 52. *The Laplacian on a closed Riemannian manifold* (M^m, g) *is self-adjoint, more precisely: Let* $f, h \in W^{2,2}(M)$*, then*

$$\langle \Delta f, h\rangle_{L^2(M)} = \langle f, \Delta h\rangle_{L^2(M)}.$$

Proof. Since the manifold is closed (compact, without boundary), the functions are weakly differentiable:

$$\begin{aligned}
\langle \Delta f, h\rangle_{L^2(M)} &= \int *(\Delta f h) \\
&= \int \frac{1}{\sqrt{g}}\partial_i\Big(\sqrt{g}g^{ij}f_{;j}\Big)h\sqrt{g}\ \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^m \\
&= -\int \Big(\sqrt{g}g^{ij}f_{;j}\Big)h_{;i}\ \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^m \\
&= -\int *\Big(f_{;j}g^{ji}h_{;i}\Big) \\
&= -\langle \mathrm{d}f, \mathrm{d}h\rangle_{L^2(T^*M)} \qquad (2.38)\\
&= \langle f, \Delta h\rangle_{L^2(M)}.
\end{aligned}$$

□

[44]This basis is named after the Polish mathematician Juliusz Paweł Schauder.

[45]This space is named after the Soviet-Russian mathematician Сергéй Львóвич Сóболев (Sergei Lvovich Sobolev).

Corollary 53. *Let* (M, g) *be a closed Riemannian manifold and* $f \in L^2(M)$*, then*

$$\int *\Delta f = 0,$$

moreover let f *be a locally non-constant eigenfunction of the Laplacian, then*

$$\int *f = 0.$$

Proof. Trivial. □

Corollary 54. *Eigenfunctions of the Laplacian on a closed Riemannian manifold with different eigenvalues are pairwise* L^2*-orthogonal.*

Proof. Let (M, g) be a closed Riemannian manifold and Let $f_1, f_2 \in L^2(M)$ with $\Delta f_1 = \lambda_1 f_1, \Delta f_2 = \lambda_2 f_2$, and $\lambda_1 \neq \lambda_2$. Since the Laplacian is self-adjoint, we have

$$\lambda_1 \langle f_1, f_2 \rangle_{L^2(M)} = \langle \Delta f_1, f_2 \rangle_{L^2(M)} = \langle f_1, \Delta f_2 \rangle_{L^2(M)} = \lambda_2 \langle f_1, f_2 \rangle_{L^2(M)},$$

which holds iff $\langle f_1, f_2 \rangle_{L^2(M)} = 0$. □

Remark 55. In (2.38) we have "discovered" $\Delta f = \mathrm{d}^* \mathrm{d} f$ where d^* is the $L^2(M)$-adjoint of d. Equation (2.38) also shows us some details of our sign convention, furthermore, let v be an $L^2(M)$-normalized eigenfunction with eigenvalue $-\lambda^2$, then

$$\|\mathrm{d}v\|^2_{L^2(T^*M)} = \langle \mathrm{d}v, \mathrm{d}v \rangle_{L^2(T^*M)} = -\langle \Delta v, v \rangle_{L^2(M)} = \lambda^2 \langle v, v \rangle_{L^2(M)} = \lambda^2,$$

hence $\|\mathrm{d}v\|_{L^2(T^*M)} = \lambda$.

Remark 56. Equation (2.36) is motivated by the following ansatz: Let $u(h, s, \phi, \tau) = X(h, s, \phi) T(\tau)$ for two functions X and T. Then

$$c^2 T \Delta X = c^2 \Delta u = \frac{\partial^2}{\partial^2 \tau} u = X \ddot{T}$$

implies for all (h, s, ϕ, τ) with $u(h, s, \phi, \tau) \neq 0$:

$$\frac{\ddot{T}}{c^2 T} = \frac{\Delta X}{X}. \tag{2.39}$$

The right hand side of (2.39) does not depend on τ and the left hand side of (2.39) does not depend on (h, s, ϕ), so both sides are (locally) constant: A real number $\lambda \in \mathbb{R}$ exists such that (locally)

$$\Delta X = \lambda X, \tag{2.40}$$

$$\ddot{T} = c^2 \lambda T. \tag{2.41}$$

The equation (2.40) is the Helmholtz equation which has no solutions for $\lambda > 0$ as stated in observation 51 and solutions of (2.41) are given by sine functions for $\lambda < 0$.

Remark 57. We also have a geometric equivalent of remark 56: Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds. Then their product

$$(M, g) = (M_1 \times M_2, g_1 + g_2)$$

is a pseudo-Riemannian manifold. Let $f_1 \in C^\infty(M_1), f_2 \in C^\infty(M_2)$, then

$$\begin{aligned} f\colon M_1 \times M_2 &\to \mathbb{R} \\ (x, y) &\mapsto f_1(x) \cdot f_2(y) \end{aligned}$$

is a smooth function on M. The same construction works for the corresponding $W^{k,p}$-spaces. Moreover, let $\Delta_{g_1}, \Delta_{g_2}$ and Δ_g be the Laplacians of $(M_1, g_1), (M_2, g_2)$ and (M,g), respectively. If $\Delta_{g_1} f_1 = \lambda_1 f_1, \Delta_{g_2} f_2 = \lambda_2 f_2$, then

$$\Delta_g f = (\lambda_1 + \lambda_2) f.$$

We now deal with a Riemannian manifold (M_1, g_1) and

$$(M_2, g_2) = (\mathbb{R}^1, -c^2 \mathrm{d}\tau \otimes \mathrm{d}\tau)$$

with the local coordinate τ. Note $-\lambda^2 = \lambda_1 \leq 0 \leq \lambda_2$. Let $\Delta_{g_1} f_1 = -\lambda^2 f_1$, then

$$\Delta_g\Big|_{(x,\tau)}\Big(f_1(x)\sin(c\lambda\tau+\tau_0)\Big) = \left(-\lambda^2 + \frac{c^2}{c^2}\lambda^2\right) f(x,\tau) = 0,$$

hence the harmonic functions on M are exactly the solutions of the wave equation on (M_1, g_1). In particular with $M_1 = \Sigma_{rt}$, we have $\Box_c = c^2\Delta_g$.

Remark 58. We observe a critical aspect of understanding observation 51: Let v_1, v_2 be two eigenfunctions: $\Delta v_1 = -\lambda_1^2 v_1, \Delta v_2 = -\lambda_2^2 v_2$ with $0 < \lambda_1^2 < \lambda_2^2$. Then the function $v = \mu_1 v_1 + \mu_2 v_2$ for any $\mu_1, \mu_2 \in \mathbb{C}$ with $\mu_1\mu_2 \neq 0$ is not an eigenfunction of the Laplacian:

$$\Delta v = -\mu_1\lambda_1^2 v_1 - \mu_2\lambda_2^2 v_2 = -\lambda_1^2 v + (\lambda_1^2 - \lambda_2^2)\mu_2 v_2.$$

This behavior is well known from linear algebra. The eigenvalue $-\lambda_2^2$ defines an overtone.

Observation 59. Note that

$$v_0 \equiv \frac{1}{\sqrt{\mathrm{vol}(\Sigma_{rt})}}$$

does not model an oscillation and that for any $(\mu_k) \in \ell^2(\mathbb{N})$:

$$\sum_{k=1}^{\infty} \mu_k v_k \in L^2(\Sigma_{rt})$$

which shows that for any sequence $(\mu_k) \in \ell^2(\mathbb{N})$ of real numbers and any sequence $(\tau_k)_{k\in\mathbb{N}}$ of real numbers the function defined by

$$u(h,s,\phi,\tau) = \sum_{k=1}^{\infty} \mu_k v_k(h,s,\phi)\sin(\lambda_k c\tau + \tau_k) \tag{2.42}$$

$$= \sum_{k=1}^{\infty} v_k(h,s,\phi)\Big(\sigma_k \sin(\lambda_k c\tau) + \gamma_k \cos(\lambda_k c\tau)\Big) \tag{2.43}$$

is a solution to the wave equation. The coefficients $(\sigma_k)_{k\in\mathbb{N}}, (\gamma_k)_{k\in\mathbb{N}}$ can be calculated with the initial data, for example at $\tau = 0$ via

$$\begin{aligned}\gamma_k &= \langle u(\,\cdot\,,\,\cdot\,,\,\cdot\,,0), v_k\rangle_{L^2(\Sigma_{rt})}\\ \sigma_k &= \frac{1}{c\lambda_k}\langle \dot{u}(\,\cdot\,,\,\cdot\,,\,\cdot\,,0), v_k\rangle_{L^2(\Sigma_{rt})},\end{aligned}$$

where in this case the notation of (2.43) leads to slightly easier formulas than (2.42), using corollary 15.

Let $\mu_1 \neq 0$. The fundamental tone has frequency $\frac{c\lambda_1}{2\pi}\cdot\mathsf{Hz}$ and the overtone's frequencies are

$$\left(\frac{c\lambda_k}{2\pi}\cdot\mathsf{Hz}\right)_{k\in\mathbb{N}_{>1}},$$

compare observation 50. The oscillation (2.42) may not cover the whole spectrum, depending on $(\mu_k)_k$. Obviously, the amplitude of the overtones of any oscillation decreases:

$$\lim_{k\to\infty}\mu_k = 0.$$

Since the k^{th} overtone satisfies

$$\sqrt[12]{2}^{\,l}\,\frac{c\lambda_1}{2\pi} = \frac{c\lambda_k}{2\pi}$$

for some $l > 0$ the k^{th} overtone is exactly

$$l\cdot\mathsf{m}_2 = (\operatorname{ls}\lambda_k - \operatorname{ls}\lambda_1)\cdot\mathsf{m}_2$$

higher than the fundamental, which is independent from c.

Observation 60. The ansatz of observation 50 can also be used on the damped wave equation. Let $\lambda \geq 0$ be a real number and $u\colon \mathbb{R}^3 \to \mathbb{R}$ be a function such that

$$\Delta u = -\lambda^2 u, \qquad \frac{\partial}{\partial\tau}u \equiv 0.$$

For any $\tau_0 \in \mathbb{R}$ the function

$$\tau \mapsto \exp(-\gamma\tau)\sin\left(\sqrt{\sigma - \gamma^2}\tau + \tau_0\right)$$

satisfies the ordinary differential equation

$$\ddot{x} + 2\gamma\dot{x} + \sigma x = 0 \tag{2.44}$$

for any $\gamma, \sigma \in \mathbb{R}$ with $\sigma > \gamma^2$, hence the function

$$\widehat{u}(h, s, \phi, \tau) := u(h, s, \phi)\exp(-\gamma\tau)\sin\left(\sqrt{c^2\lambda^2 + \sigma}\tau + \tau_0\right)$$

satisfies the *damped wave equation*

$$c^2\Delta\widehat{u} = \left(\frac{\partial^2}{\partial^2\tau} + 2\gamma\frac{\partial}{\partial\tau} + \gamma^2 + \sigma\right)\widehat{u} \tag{2.45}$$

for $0 \leq c^2\lambda^2 + \sigma$. As expected this solution equals (2.36) for $\gamma = \sigma = 0$. The frequency of the oscillation is

$$\frac{\sqrt{c^2\lambda^2 + \sigma}}{2\pi} \cdot \mathsf{Hz} = \frac{c\lambda}{2\pi}\sqrt{1 + \frac{\sigma}{c^2\lambda^2}} \cdot \mathsf{Hz} \tag{2.46}$$

as long as τ models the time in seconds. We have

$$\lim_{\tau\to\infty} \widehat{u}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\tau) = 0$$

iff $\gamma > 0$. In this case, the half-life of the peak amplitude is ${}^{\log 2}/_{\gamma}$. If $\sigma \neq 0$, then the frequencies defined by the damped wave equation are

$$\frac{1}{2}\mathrm{ls}\left(1 + \frac{\sigma}{c^2\lambda^2}\right)$$

semitones higher compared to the undamped wave equation, compare (2.46). This difference also depends on λ.

For $\sigma \leq \gamma^2$ the solutions of (2.44) are

$$\tau \mapsto a\exp\left(\left(-\gamma - \sqrt{\gamma^2 - \sigma}\right)\tau\right) + b\exp\left(\left(-\gamma + \sqrt{\gamma^2 - \sigma}\right)\tau\right),$$

for any $a, b \in \mathbb{R}$ which obviously does not model a damped oscillator. Given a linear ordinary differential operator D with real, constant coefficients, it is well known that the differential equation $Dx \equiv 0$ admits a damped oscillation iff the characteristic polynomial of D has complex roots. Another generalization is based on a differential operator D of τ: Let T be an eigenfunction of D with eigenvalue $-c^2\lambda^2$ and X be an eigenfunction of Δ with eigenvalue $-\lambda^2$, then the function $u(h, s, \phi, \tau) = X(h, s, \phi)T(\tau)$ is a solution of the partial differential equation

$$c^2 \Delta u = Du.$$

Example 61. We illustrate the fundamental tone and overtones of figure 9 in figure 12. We refer to [TM08, section 15-2: harmonic sound waves, section 15-3: wave intensity]. Let $k \in \mathbb{Z}_{>0}$. The oscillation of the fundamental tone is exactly $s_0 = g_1$ and the oscillation of the k^{th} overtone is

$$s_k \colon \tau \mapsto \frac{1}{4\sqrt[10]{10}k^2} \cos(k\pi\tau),$$

hence the k^{th} overtone is

$$o_k = \left(\operatorname{ls}(k\pi) - \operatorname{ls}\frac{\pi}{4} \right) \cdot \mathsf{m}_2 = (24 + \operatorname{ls} k) \cdot \mathsf{m}_2$$

higher than the fundamental tone, compare (2.101), (2.101), and table 9. Let p_0 the pressure amplitude of the fundamental tone and p_k the pressure amplitude of the k^{th} overtone: We observe

$$p_0 = \xi \cdot 1 \cdot \frac{\pi}{4},$$
$$p_k = \xi \cdot \frac{1}{4\sqrt[10]{10}k^2} \cdot k\pi$$

for some ξ. Therefore the k^{th} overtone is

$$d_k = -10 \frac{\log\left(\frac{p_k}{p_0}\right)^2}{\log 10} \cdot \mathsf{dB} = \left(\frac{20}{\log 10} \log k + 2 \right) \cdot \mathsf{dB}$$

quieter than the fundamental, compare table 9.

The decrease of amplitude is independent of the frequency (2.46) of the oscillation according to the damped wave equation (2.45) and its solution $\widehat{u}$.

Observation 62. The ansatz as in remark 56 obviously does not work when c is not constant, since then the function T would not be independent from the arguments (h, s, ϕ). Naturally a non-constant c is not compatible with the ansatz as in remark 57 either.

For this paragraph, we assume that c is locally constant Lebesgue-almost everywhere and independent from τ. Let u be a smooth eigenfunction of the Laplacian; For a constant $\lambda \in \mathbb{R}$ we have

$$\Delta u = -\lambda^2 u, \qquad \frac{\partial}{\partial \tau} u \equiv 0.$$

Then for any $\tau_0 \in \mathbb{R}$, the function

$$\begin{aligned} \widehat{u} \colon \Sigma_{rt} \times \mathbb{R} &\to \mathbb{R} \\ (h, s, \phi, \tau) &\mapsto u(h, s, \phi) \cdot \sin\Big(c(h, s, \phi) \cdot \lambda\tau + \tau_0\Big) \end{aligned}$$

is a solution of the wave equation and smooth Lebesgue-almost everywhere. An interpretation of this result is a tarnish spot on a bell. Assuming that there are only a few small tarnish spots and c behaves as written above, the tarnish spots do not have a major influence on the tone as the oscillations do not change on the majority of the bell.

Observation 63. From the ansatz of (2.36), we see that all overtones of two different bells, which differ only in terms of the constants c_1, c_2 and thus share the same shape and size, differ only by the factor ${}^{c_1}/_{c_2}$ or ${}^{c_2}/_{c_1}$, respectively. This corresponds to the factor B of to the bell formula (2.7) and the constant γ as in observation 34.

More generally, this property holds, if the eigenvalues and eigenfunctions of the Laplacian are the same.

Observation 64. Scaling a bell by a constant factor $\gamma > 0$ yields a different metric $\widetilde{g}$, we conclude $\widetilde{\Gamma}_r = \gamma\Gamma_r, \widetilde{t} = \gamma t, \widetilde{G} = G$ from (2.27) and thus $\widetilde{g} = \gamma^2 g, \det \widetilde{g} = \gamma^6 \det g$, hence scaling a bell results in a trivial conformal change of the metric.

Let u be an eigenfunction to the eigenvalue $-\lambda^2$ of the Laplacian Δ, thus

$$\begin{aligned}\widetilde{\Delta}u &= \frac{1}{\sqrt{\widetilde{g}}}\partial_i\Big(\sqrt{\widetilde{g}}\,\widetilde{g}^{ij}\partial_j u\Big)\\ &= \frac{1}{\sqrt{\gamma^6 g}}\partial_i\left(\sqrt{\gamma^6 g}\frac{g^{ij}}{\gamma^2}\partial_j u\right)\\ &= \frac{1}{\gamma^2}\Delta u\\ &= -\frac{\lambda^2}{\gamma^2}u\end{aligned}$$

follows from (2.33), hence

$$\widehat{u}(h,s,\phi,\tau) := u(h,s,\phi)\cdot\sin\left(\frac{\lambda c}{\gamma}\tau+\tau_0\right)$$

is a solution of the wave equation of the scaled bell for any $\tau_0\in\mathbb{R}$, compare (2.36). This result is compatible with observation 24.

This shows that the characteristic sound of a bell is independent from scaling, since all eigenvalues are scaled by the same factor, hence the ratio of two eigenvalues is independent from scaling: Every partial tone is lowered by $\operatorname{ls}\gamma\cdot\mathsf{m}_2$.

Observation 65. To get a solution of the wave equation, we need to discuss the initial value problem. Let u be a solution of the Helmholtz equation modeling a $[m,n]$ node. From section 2.6 we see

$$\lim_{h\searrow 0} u(h,\,\cdot\,,\,\cdot\,)\equiv 0, \tag{2.47}$$

$$\lim_{h\searrow 0}\partial_h u(h,\,\cdot\,,\,\cdot\,)\equiv 0, \tag{2.48}$$

$$u(h,s,\,\cdot\,)\in C^1\Big(S^1\Big), \tag{2.49}$$

$$\#\Big(u(h,s,\,\cdot\,)\Big)^{-1}(0)=m, \tag{2.50}$$

$$\#\Big(u(\,\cdot\,,s,\phi)\Big)^{-1}(0)=n. \tag{2.51}$$

Different from a drum, we do not have the condition

$$\lim_{h \nearrow H} u(h, \,\cdot\,, \,\cdot\,) = 0,$$

compare figure 11. This condition makes computing the oscillations of a drum a trivial task since it would determine λ as in (2.88).

Since at $h = 0$ the casting is fixed to the handle, there are no oscillations, hence (2.47) and (2.48). The equation (2.49) is obvious from symmetry. Since the number of meridian nodes is m, equation (2.50) is given. Since the number of circular nodes is n, equation (2.51) is given. Note that (2.50) and (2.51) do not admit a constant solution.

Remark 66. We now have a basic understanding of the importance of the eigenvalues of the Laplacian: They determine the sound a bell makes. The well-known and related question "Can one hear the shape of a drum?" leads to a well-known problem in mathematics. This brings up questions:

(a) To what extent do the eigenvalues determine the shape of a bell?

(b) How stable are the eigenvalues with respect to small perturbations of the shape?

We will have a look on the geometry to get a basic understanding but will not answer these questions. The publications [Cru03] and notably [Ber03, chapter 9] give a short introduction to this topic.

We first discuss the most trivial example of the wave equation, the d-dimensional solid brick, compare figure 15a:

Example 67. Let $H_1, \ldots, H_d \in \mathbb{R}_{>0}, M = \prod_{k=1}^{d} [0, H_k] \subsetneq \mathbb{R}^d$ and

$$\begin{aligned} \Sigma \colon [0,1]^d &\to M \\ (x^1, \ldots, x^d) &\mapsto (H_1 x^1, \ldots, H_d x^d), \end{aligned}$$

we then calculate the metric as the diagonal matrix

$$g(x) = \begin{pmatrix} H_1^2 & & \\ & \ddots & \\ & & H_d^2 \end{pmatrix}$$

and the Laplacian

$$\Delta u = \frac{1}{\sqrt{g}} \partial_i \Big(\sqrt{g} g^{ij} \partial_j u \Big) = \sum_{k=1}^{d} \frac{u_{;kk}}{H_k^2},$$

of a function u, hence for $s_1, \ldots, s_d \in \{0,1\}, \lambda_1, \ldots, \lambda_d \in \mathbb{R}_{\geq 0}, \tau_0 \in \mathbb{R}$, and $\lambda = \sqrt{\sum_{k=1}^d \lambda_k^2}$ the function

$$u(x,\tau) = \sin\left(c\lambda\tau + \tau_0\right) \prod_{k=1}^{d} \Big(s_k \cos(\lambda_k H_k x_k) + (1 - s_k) \sin(\lambda_k H_k x_k) \Big)$$

is a solution of the wave equation. With the boundary condition $u|_{\partial M \times \mathbb{R}} \equiv 0$, we consider

$$u(x,\tau) = C \sin\left(\tau_0 + \tau 2\pi c \sqrt{\sum_{k=1}^{d} \left(\frac{\mu_k}{H_k}\right)^2} \right) \prod_{k=1}^{d} \sin(2\pi\mu_k x_k) \tag{2.52}$$

with $\mu_k \in \mathbb{N}$ and $C \in \mathbb{R}$ as a solution of the wave equation, hence the frequencies of oscillation are given by

$$f_{[\mu_1,\ldots,\mu_d]} = c \sqrt{\sum_{k=1}^{d} \left(\frac{\mu_k}{H_k}\right)^2} \cdot \mathsf{Hz} \tag{2.53}$$

compare observation 50. The frequencies $f_{\mu_1,\ldots,\mu_d}$ are dependent on the constant c and on the size of the brick. Note that u has $(\mu_k - 1)$ nodes in the k^{th} dimension. We conclude from (2.53), that bigger bricks have lower tones and scaling only one dimension of the brick for $d > 1$ does not scale the frequencies, the change is not that trivial. We also have d notes which corresponds to the number of dimensions whereas the bell formula only covers two dimensions of a three dimensional object.

Another interpretation with appropriate boundary conditions are bell plates. We also note that the solid brick can also seen as a product of one-dimensional Riemannian manifolds as in remark 57.

To continue with more useful examples, we first need to introduce some background.

Observation 68. Let $a_\alpha\colon \Sigma_{rt} \to \mathbb{R}$ be functions where $\alpha \in \mathbb{N}_0^4$ is a multi-index for (x, τ) and let

$$D := \sum_{|\alpha| \leq n} a_\alpha \partial_\alpha$$

be a linear partial differential operator or order $n \in \mathbb{N}_0$, which applies to both, the wave equation and the damped wave equation. Let $u\colon \mathbb{R} \to \mathbb{C}$ a complex solution of $Du = 0$. Since the operators $\Re$ and $\Im$ are smooth, the real functions $\Re u, \Im u$ are also solutions of $Du = 0$, since

$$0 = Du = D(\Re u + i\Im u) = D\Re u + iD\Im u,$$

hence $\operatorname{span}_{\mathbb{C}}\{\Re u, \Im u\} \subseteq \ker D$. In particular, we recall EULER's formula[46]

$$\exp z = (\cos \Im z + i \sin \Im z) \exp \Re z$$

for any $z \in \mathbb{C}$.

The following statement sums up some well known results on Fourier-series applied on a special case.

Corollary 69. *Let $H > 0$ and $f \in C^1\Big([0, H], \mathbb{R}\Big)$ with $f'(0) = 0 = f'(H)$ and $(c_k)_{k \in \mathbb{N}_0}$ the sequence defined by*

$$c_k = \frac{2}{H} \int_0^H f(\xi) \cos \frac{\pi k \xi}{H} \, \mathrm{d}\xi.$$

Then $(c_k) \in \ell^2(\mathbb{N}_0)$ and

$$f(\xi) \equiv \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos \frac{\pi k \xi}{H}$$

for all $\xi \in [0, H]$.

[46] LEONHARD EULER was an important mathematician from the old Swiss Confederacy.

Proof. We define the continuation

$$\widetilde{f} = f \circ |\cdot|\Big|_{[-H,H]}.$$

The restrictions ensure that the function $\widetilde{f} \in C^1\Big([-H,H]\Big)$ is even and satisfies

$$\begin{aligned}
\lim_{\xi \searrow -H} \widetilde{f}(\xi) &= f(H) = \lim_{\xi \nearrow H} \widetilde{f}(\xi),\\
\lim_{\xi \searrow -H} -\widetilde{f}'(\xi) &= f'(H) = \lim_{\xi \nearrow H} \widetilde{f}'(\xi),
\end{aligned}$$

therefore the Fourier-series $\widehat{f}$ of $\widetilde{f}$ converges uniformly to $\widetilde{f}$, hence $\widehat{f}\Big|_{[0,H]} \equiv f$. Since $\widetilde{f}$ is even, the coefficients of sine vanish and the coefficients of cosine are

$$\begin{aligned}
c_k &= \frac{2}{2H} \int_{-H}^{H} \widetilde{f}(\xi) \cos \frac{2\pi k \xi}{2H} \, \mathrm{d}\xi\\
&= \frac{2}{H} \int_{0}^{H} f(\xi) \cos \frac{\pi k \xi}{H} \, \mathrm{d}\xi.
\end{aligned}$$

Since $\widetilde{f} \in C^1\Big([-H,H]\Big)$ we conclude $\widetilde{f} \in L^2\Big(S^1\Big)$, and so

$$\frac{\Big\|(c_k)\Big\|^2_{\ell^2(\mathbb{N}_0)}}{2} - \frac{c_0^2}{4} = \frac{c_0^2}{4} + \sum_{k=1}^{\infty} \frac{c_k^2}{2} = \frac{1}{2H} \int_{-H}^{H} \widetilde{f}^2(\xi) \, \mathrm{d}\xi \in \mathbb{R}. \qquad \square$$

Remark 70. We can also use a different function as a differentiable continuation in corollary 69 to generalize the two restrictions $f'(0) = 0, f'(H) = 0$ to $f'(0), f'(H) \in \mathbb{R}$, which will give us more cumbersome coefficients and non-vanishing sine-coefficients. Among several options such as polynomials or splines, a reasonable choice is

$$\begin{aligned}
\widetilde{f}\Big|_{[-H,0]}(\xi) = \frac{f(0) + f(H)}{2} &+ \frac{f(0) - f(H)}{2} \cos \frac{\pi \xi}{H}\\
&+ \frac{\Big(f'(0) - f'(H)\Big) H}{2\pi} \sin \frac{\pi \xi}{H} + \frac{\Big(f'(0) + f'(H)\Big) H}{4\pi} \sin \frac{2\pi \xi}{H},
\end{aligned}$$

which ensures $\widetilde{f} \in L^2\Big(S^1\Big) \cap C^1\Big(S^1\Big)$.

Recall two classic examples from calculus:

Example 71. Let

$$\begin{aligned} M &= \Big\{x \in \mathbb{E}^2 \Big| 0 \le |x| \le 1\Big\}, \\ N &= \Big\{x \in \mathbb{E}^2 \Big| 0 < |x| < 1\Big\}, \end{aligned}$$

and g be the Euclidean Riemannian metric restricted to M, hence $(N, g|_N)$ is a Riemannian submanifold of (M, g). Let

$$\begin{aligned} f &= \sqrt{-\log|\,\cdot\,|^2}\Bigg|_N, \\ h &\colon N \to \mathbb{R} \\ (x, y) &\mapsto \frac{x^2}{x^2 + y^2}. \end{aligned}$$

Then for any $(r, \phi) \in (0, 1) \times \mathbb{R}$ we have

$$\begin{aligned} h(r\cos\phi, r\sin\phi) &= \cos^2\phi, \qquad (2.54) \\ \|h\|^2_{L^2(N)} &= \frac{3}{8}\pi, \end{aligned}$$

and

$$\begin{aligned} f(r\cos\phi, r\sin\phi) &= \sqrt{-2\log r}, \qquad (2.55) \\ \|f\|^2_{L^2(N)} &= -2\int_0^1\int_0^{2\pi} \log r\ r\mathrm{d}\phi\mathrm{d}r \\ &= \pi \lim_{\xi \searrow 0}\left[r^2 - \frac{2\log r}{r^{-2}}\right]_\xi^1 \\ &= \pi, \\ \nabla f|_{(r\cos\phi, r\sin\phi)} &= \frac{-1}{r\sqrt{-2\log r}}(\cos\phi, \sin\phi), \qquad (2.56) \end{aligned}$$

hence $f, h \in L^2(N)$ and $f, h \in C^\infty(N)$. From (2.55) we deduce

$$\lim_{x \to 0} f(x) = \infty,$$

hence a continuous continuation of f on M does not exist. Even though we see

$$\lim_{x \to y} f(x) = 0,$$

from (2.55) for any $y \in S^1 = \partial M$, we see that f does not admit a differentiable continuation on $N \cup S^1 \subseteq \mathbb{E}^2$, compare (2.56). From (2.54) we deduce that h does not admit a continuous continuation on M.

Example 72. Let

$$\begin{aligned} M &= (-\pi, \pi] \subseteq \mathbb{E}^1, \\ N &= (0, \pi] \subseteq M, \end{aligned}$$

and g be the Euclidean Riemannian metric restricted to M, hence $(N, g|_N)$ is a Riemannian submanifold of (M, g). We consider the canonical inner products

$$\begin{aligned} \langle\ \cdot\ ,\ \cdot\ \rangle_{L^2(M)} \colon L^2(M) \times L^2(M) &\to \mathbb{C} \\ (f_1, f_2) &\mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(x) \overline{f_2(x)}\ \mathrm{d}x, \\ \langle\ \cdot\ ,\ \cdot\ \rangle_{L^2(N)} \colon L^2(N) \times L^2(N) &\to \mathbb{C} \\ (f_1, f_2) &\mapsto \frac{1}{\pi} \int_{0}^{\pi} f_1(x) \overline{f_2(x)}\ \mathrm{d}x \end{aligned}$$

on the corresponding L^2-spaces. Let

$$\begin{aligned} e_k \colon M &\to \mathbb{C} \\ x &\mapsto \exp(kxi), \\ \mathcal{B}_M &= (e_k)_{k \in \mathbb{Z}}, \\ \mathcal{B}_{\widetilde{N}} &= \left(e_k|_N \right)_{k \in \mathbb{Z}}, \\ \mathcal{B}_N &= \left(e_{2k}|_N \right)_{k \in \mathbb{Z}}. \end{aligned}$$

A orthonormal Schauder basis of $L^2(M)$ is given by $\mathcal{B}_M$, whereas a orthonormal Schauder basis of $L^2(N)$ is given by B_N. The set $\mathcal{B}_{\widetilde{N}}$ is neither an orthogonal set in $L^2(N)$, nor a Schauder basis of $L^2(N)$, but a spanning set.

There is one major issue: We can naturally define $e_k\colon x \mapsto \exp(kxi)$ on $\mathbb{R}$. Then $e_k(x) = e_k(x+\pi)$ iff k is even, hence e_k is only continuous on $S^1 \cong \mathbb{R}/(\pi\,\mathbb{Z})$ iff k is even.

Finally, we note

$$\Delta e_k = -k^2 e_k.$$

Observation 73. Let $(N, g|_N)$ be a trivial Riemannian submanifold of the compact Riemannian manifold (M, g). Then the inclusion map

$$\begin{aligned} \iota\colon L^2(N) &\hookrightarrow L^2(M) \\ u &\mapsto \left(x \mapsto \begin{cases} u(x), & x \in N \\ 0, & x \notin N \end{cases} \right) \end{aligned}$$

preserves the $L^2(N)$-norm

$$\|u\|_{L^2(N)} = \|\iota \circ u\|_{L^2(M)}$$

for all $u \in L^2(N)$. Sometimes the notion $\iota \circ u = \chi_N \cdot u = \mathbb{1}_N \cdot u$ is used. Hence the inner product is preserved, too. This Identification makes $L^2(N)$ a Hilbert subspace of $L^2(M)$. Let $(u, v) \in L^2(N) \times L^2(M)$, then

$$\langle \iota(u), v \rangle_{L^2(M)} = \langle u, v|_N \rangle_{L^2(N)}, \tag{2.57}$$

hence an orthogonal basis of $L^2(M)$ gives a spanning set of $L^2(N)$ via restriction and $\iota(u)$ has a unique representation in the orthogonal basis, compare example 72. If $u \in C^\alpha(N)$ another continuation $\widetilde{\iota}(u) \in C^\alpha(M)$ might be appropriate, provided its existence, compare example 71. The proof of corollary 69 and remark 70 give examples on simple domains. These inclusion maps $\widetilde{\iota}$ may make $C^\alpha(N)$ a subspace but may not preserve the $L^2(N)$-norm, hence nor the $L^2(N)$ inner product.

Considering observation 73 and example 67 the considerations of specific geometric shapes of bells might be redundant, but specific calculations give more useful results, compare observation 51 and example 72.

Lemma 74. *Let $p_0, p_1, p_2, h \in C(\mathbb{R})$ be continuous functions and let $y \in C^\infty(\mathbb{R})$ be a smooth function. If f is a solution of the ordinary differential equation*

$$p_2 f'' + p_1 f' + p_0 f = h,$$

then $F = f \circ y$ is a solution of the ordinary differential equation

$$y' p_2 \circ y F'' + \left(\left(y'\right)^2 p_1 \circ y - y'' p_2 \circ y \right) F' + \left(y'\right)^3 p_0 \circ y F = \left(y'\right)^3 h \circ y,$$

in particular for $y' \neq 0$ this is equation is equivalent to

$$\frac{p_2 \circ y}{\left(y'\right)^2} F'' + \left(\frac{p_1 \circ y}{y'} - \frac{y'' p_2 \circ y}{\left(y'\right)^3} \right) F' + p_0 \circ y F = h \circ y.$$

Proof. A simple calculation shows the statement. □

Definition 75. Let $m \in \mathbb{C}$. The *Bessel equation of order m* is the linear, homogeneous ordinary differential equation of order 2

$$h^2 \frac{\mathrm{d}^2 y}{\mathrm{d}^2 h}(h) + h \frac{\mathrm{d} y}{\mathrm{d} h}(h) + (h^2 - m^2) y(h) = 0. \tag{2.58}$$

Lemma 76. *Let $y_1 \in C^2(D_1)$ be a solution of the Bessel equation of order m on some domain $D_1 \subseteq \mathbb{R}$.*

(a) If $m \neq 0$ and y_1 is defined on 0, then $y_1(0) = 0$.

(b) The vector space of solutions of the Bessel equation of order m has dimension 2.

(c) The function

$$y_2 := \frac{2}{\pi} y_1 \int \frac{1}{\xi y_1^2(\xi)} \, \mathrm{d}\xi$$

is a solution of the Bessel equation of order m. The domain $D_2 \subseteq \mathbb{R}$ of y_2 likely does not contain the origin. The functions y_1, y_2 are linear independent and form a basis on $D_1 \cap D_2$ of the vector space of solutions of the Bessel equation of order m.

(d) Let f be a smooth function without a horizontal tangent. Then $\widetilde{y} = y_1 \circ f$ (defined on a proper domain) is a smooth solution of

$$\left(\frac{f}{f'}\right)^2 \widetilde{y}'' + \left(\frac{f}{f'} - \frac{f^2 f''}{f'^3}\right) \widetilde{y}' + \left(f^2 - m^2\right) \widetilde{y} = 0. \tag{2.59}$$

Proof. We use well-known results of the theory of ordinary differential equations.

The equation $m y_1(0) = 0$ is obtained by substituting $h = 0$ in (2.58).

For $h \neq 0$, (2.58) is equivalent to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}^2 h}(h) + \frac{1}{h} \frac{\mathrm{d}y}{\mathrm{d}h}(h) + \left(1 - \frac{m^2}{h^2}\right) y(h) = 0,$$

which is the standard notation for a linear ordinary differential equation of order 2; The coefficients are $a_1(h) = {}^1\!/_h$ and $a_0(h) = 1 - {}^{m^2}\!/_{h^2}$. The dimension of the vector space of solutions is the order of the ordinary differential equation.

The d'Alembert reduction gives the second solution y_2 and its independence.

A simple calculation with lemma 74 shows the last statement from (2.58).□

Recall:

Proposition 77. *Let $m \in \mathbb{Z}$ be an integer. The Bessel function*[47] *of first kind and order m is defined as*

$$\begin{aligned} J_m \colon \mathbb{R} &\to \mathbb{R} \\ h &\mapsto \frac{1}{\pi} \int_0^{\pi} \cos\left(m\xi - h \sin \xi\right) \, \mathrm{d}\xi, \end{aligned} \tag{2.60}$$

compare figure 13.

Let $(j_{mk})_{k\in\mathbb{N}} = \left(j^0_{mk}\right)_{k\in\mathbb{N}}$ be the strictly increasing sequence of roots of $J_m|_{\mathbb{R}_{>0}}$ and let $\left(j^1_{mk}\right)_{k\in\mathbb{N}}$ be the strictly increasing sequence of critical points of $J_m|_{\mathbb{R}_{>0}}$, hence both sequences are defined by

$$\begin{aligned} k &= \#\{x \in (0, j_{mk}] | J_m(x) = J_m(j_{mk}) = 0\}, \\ k &= \#\left\{x \in \left(0, j^1_{mk}\right] \middle| J'_m(x) = J'_m\left(j^1_{mk}\right) = 0\right\}, \end{aligned}$$

see table 10a for some examples. Let $H > 0$ and

$$\begin{aligned} \langle \,\cdot\, , \,\cdot\, \rangle_{\widehat{g}} \colon L^2(0, H) \times L^2(0, H) &\to \mathbb{C} \\ (f_1, f_2) &\mapsto \int_0^H f_1(h) \overline{f_2(h)} h \, \mathrm{d}h \end{aligned}$$

be a Hermitian form.

(a) *The Bessel function J_m is a smooth solution of the Bessel equation of order m.*

(b) *The sequences $(j_{mk})_{k\in\mathbb{N}}$ and $\left(j^1_{mk}\right)_{k\in\mathbb{N}}$ as stated above, exist.*

(c) *The Bessel functions J_m satisfy the recursion identities*

$$J_m(h) = J_{-m}(-h) = (-1)^m J_{-m}(h) = (-1)^m J_m(-h), \tag{2.61}$$

$$\frac{\mathrm{d}}{\mathrm{d}h} J_m(h) = \frac{1}{2}\Big(J_{m-1}(h) - J_{m+1}(h)\Big), \tag{2.62}$$

$$J_m(h) = \lim_{\xi \to h} \left(\frac{2(m-1)}{\xi} J_{m-1}(\xi) - J_{m-2}(\xi) \right), \tag{2.63}$$

[47] The ordinary differential equation and its solutions are named after the German mathematician Friedrich Wilhelm Bessel.

for all $h \in \mathbb{R}$.

(d) *All* J_m *are determined by* J_0 *and* J_0'.

(e) *Let* $v_k := h \mapsto J_m\Big(j_{mk}\frac{h}{H}\Big)$, *then the orthogonality property*

$$\langle v_k, v_l\rangle_{\widehat{g}} = \delta_{lk}\frac{H^2}{2}\Big(J_m'(j_{ml})\Big)^2 \tag{2.64}$$

$$= \delta_{lk}\frac{H^2}{2}\Big(J_{m+1}(j_{ml})\Big)^2, \tag{2.65}$$

which is zero iff $k \neq l$, *holds.*

(f) *Let* $u_k := h \mapsto J_m\Big(j^1_{mk}\frac{h}{H}\Big)$, *then the orthogonality property*

$$\langle u_k, u_l\rangle_{\widehat{g}} = -\delta_{lk}\frac{H^2}{2}J_m\Big(j^1_{ml}\Big)J_m''\Big(j^1_{ml}\Big) \tag{2.66}$$

$$= \delta_{lk}\frac{H^2}{2}\left(1-\left(\frac{m}{j^1_{ml}}\right)^2\right)\left(J_m\Big(j^1_{ml}\Big)\right)^2, \tag{2.67}$$

which is zero iff $k \neq l$, *holds.*

(g) *The sequence* $\Big(j^1_{mk}\Big)_{k\in\mathbb{N}}$ *is exactly the sequence of extremum points of* $J_m\big|_{(0,\infty)}$. *The point* j^1_{mk} *is local maximum iff* $J_m\Big(j^1_{mk}\Big) > 0$.

(h) *For all* $m \in \mathbb{N}_0$ *and all* $k \in \mathbb{N}$:

$$m < j^1_{mk}, \tag{2.68}$$

$$0 \neq J_m\Big(j^1_{mk}\Big), J_m'(j_{mk}), J_m''\Big(j^1_{mk}\Big), J_{m+1}(j_{mk}). \tag{2.69}$$

(i) *Similarly to* $\Big(j^1_{mk}\Big)_{k\in\mathbb{N}}$, *let* $\Big(j^2_{mk}\Big)_{k\in\mathbb{N}}$ *be the sequence of roots of* J_m'', *hence*

$$k = \#\Big\{x \in \Big(0, j^2_{mk}\Big]\Big| J_m''(x) = J_m''(j^2_{mk}) = 0\Big\}.$$

The sequence $\Big(j^2_{mk}\Big)_{k\in\mathbb{N}}$ *exists.*

(j) *The function J_m is given by the power series*

$$J_m(h) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(l+m)!} \left(\frac{h}{2}\right)^{2l+m} \tag{2.70}$$

at 0 *with radius of convergence* ∞.

(k) *For an integer k with $0 \leq k \leq m$, the k^{th} derivative of J_m at* 0 *is*

$$J_m^{(k)}(0) = \delta^{mk} 2^{-m}, \tag{2.71}$$

note $J_m^{(0)} = J_m$.

Proof. Step 1. Noting that differentiation under the integral sign is valid with respect to Lebesgue-integration, the functions J_m are obviously smooth. With

$$\begin{aligned} \chi_m(h,\xi) &= \cos(m\xi - h\sin\xi), \\ \varsigma_m(h,\xi) &= \sin(m\xi - h\sin\xi), \end{aligned}$$

we compute

$$\begin{aligned} -2\int h^2 \frac{\partial^2 \chi_m}{\partial^2 h}(h,\xi) + h\frac{\partial \chi_m}{\partial h}(h,\xi) + (h^2 - m^2)\chi_m(h,\xi)\ \mathrm{d}\xi \\ = h\varsigma_{m+1}(h,\xi) + h\varsigma_{m-1}(h,\xi) + 2m\varsigma_m(h,\xi), \end{aligned}$$

hence J_m solves (2.58).

Step 2 (roots of J_m and J_m'). For the existence of the sequences $(j_{mk})_{k\in\mathbb{N}}$ and $\left(j_{mk}^1\right)_{k\in\mathbb{N}}$ as stated, we refer to [Bow58, Chapter §96].

Step 3 (recursion identities). The equations (2.61), (2.62) and (2.63) follow from (2.60) and (2.58).

Step 4. Since (2.61) and (2.62), we have $J_1 = -J_{-1} = -J_0'$. Since (2.63), J_0 and J_1 determine $(J_m)_{m\in\mathbb{N}_0}$. Since (2.61), all $(J_m)_{m\in\mathbb{N}_0}$ determine $(J_m)_{m\in\mathbb{Z}}$.

Step 5 (Sturm-Liouville theory). For the orthogonality properties we note, that the Bessel equation is a standard example of the *Sturm-Liouville theory*:[48] Let $a < b$ be two real numbers and

$$B(h) = p_2(h)\frac{\mathrm{d}^2}{\mathrm{d}^2 h} + p_1(h)\frac{\mathrm{d}}{\mathrm{d}h} + p_0(h) \tag{2.72}$$

be a differential operator with $p_0, p_1, p_2 \in C^2(a,b)$ and $\dot{p}_2 = p_1$. For any functions $u, v \in C^2(\mathbb{R})$ with

$$\Big[p_2(v\dot{u} - \dot{v}u)\Big]_a^b = 0 \tag{2.73}$$

we compute

$$\begin{aligned}\langle Bu, v\rangle_{L^2(a,b)} &= \Big[p_2(v\dot{u} - \dot{v}u) + (p_1 - \dot{p}_2)uv\Big]_a^b \\ &\quad + \langle p_2\ddot{v} + (2\dot{p}_2 - p_1)\dot{v} + (\ddot{p}_2 - \dot{p}_1 + p_0)v, u\rangle_{L^2(a,b)} \\ &= \langle Bv, u\rangle_{L^2(a,b)}.\end{aligned} \tag{2.74}$$

If B is not self-adjoint with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(a,b)}$, then with

$$\widehat{g}(h) = -\frac{1}{p_2(h)}\exp\int\frac{p_1(h)}{p_2(h)}\,\mathrm{d}h$$

the operator

$$\widehat{B} = \widehat{g}B$$

is and $\Big(\widehat{g}p_2\Big)' = \widehat{g}p_1$. Note that $\dot{p}_2 \propto p_1$ implies const. $= \widehat{g}$.

Step 6 (Sturm-Liouville applied). Let $(a,b) = (0,H)$. The functions

$$p_2(h) = -1, \qquad p_1(h) = -\frac{1}{h}, \qquad p_0(h) = \frac{m^2}{h^2}$$

[48] The theory is named after the French mathematicians Jacques Charles François Sturm and Joseph Liouville. Comparing (2.33), the presence of Sturm-Liouville is not surprising.

define a differential operator B as in (2.72) with

$$\widehat{g}(h) = h, \tag{2.75}$$

$$BJ_m(\theta\ \cdot\ ) = \theta^2 J_m(\theta\ \cdot\ ), \tag{2.76}$$

$$\begin{aligned}\langle BJ_m(\theta\ \cdot\ ), J_m(\xi\ \cdot\ )\rangle_{\widehat{g}} &= \langle J_m(\theta\ \cdot\ ), BJ_m(\xi\ \cdot\ )\rangle_{\widehat{g}} &(2.77)\\ &= \theta^2 \langle J_m(\theta\ \cdot\ ), J_m(\xi\ \cdot\ )\rangle_{\widehat{g}} &(2.78)\\ &= \xi^2 \langle J_m(\theta\ \cdot\ ), J_m(\xi\ \cdot\ )\rangle_{\widehat{g}}, &(2.79)\end{aligned}$$

for any $\theta, \xi \in \mathbb{R}$ as long as (2.73) is satisfied. From (2.75), we see that B is self-adjoint with respect to $\langle\ \cdot\ ,\ \cdot\ \rangle_{\widehat{g}}$. From (2.74) and (2.76) we deduce with $\Theta = J_m(\theta\ \cdot\ )$ and $\Xi = J_m(\xi\ \cdot\ )$

$$\langle \Theta, \Xi \rangle_{\widehat{g}} = \frac{\left[\widehat{g}p_2\Big(\Xi\theta J'_m(\theta\ \cdot\ ) - \xi J'_m(\xi\ \cdot\ )\Theta\Big)\right]_a^b}{\theta^2 - \xi^2}. \tag{2.80}$$

Step 7. The equations (2.78) and (2.79) imply (2.65) and (2.67) for $l \neq k$ without any computations, since the condition (2.73) is satisfied for both families of functions.

Step 8. Contrary, let $l = k$. Using L'Hôpital's rule on (2.80) with $v_k = \Theta = v_l$, the equation (2.65) follows with (2.58), (2.62) and (2.63), since

$$\begin{aligned}&\lim_{\theta\to\xi} \frac{\widehat{g}p_2\Big(\Xi\theta J'_m(\theta\ \cdot\ ) - \xi J'_m(\xi\ \cdot\ )\Theta\Big)}{\theta^2 - \xi^2}(h)\\ &\quad = -h\frac{J_m(\xi h)\Big(J'_m(\xi h) + h\xi J''_m(\xi h)\Big) - h\xi\Big(J'_m(\xi h)\Big)^2}{2\xi}\end{aligned} \tag{2.81}$$

and in particular

$$\begin{aligned}&\lim_{\theta\to\xi} \frac{\widehat{g}p_2\Big(\Xi\theta J'_m(\theta\ \cdot\ ) - \xi J'_m(\xi\ \cdot\ )\Theta\Big)}{\theta^2 - \xi^2}\Bigg|_0^H\\ &\quad = H\frac{H\xi\Big(J'_m(\xi H)\Big)^2 - J_m(\xi H)\Big(J'_m(\xi H) + H\xi J''_m(\xi H)\Big)}{2\xi}.\end{aligned} \tag{2.82}$$

In a similar manner (2.67) follows with $u_k = \Theta = u_l$.

Step 9. The equation (2.66) implies

$$0 \neq \operatorname{sgn} J_m\left(j^1_{mk}\right) = -\operatorname{sgn} J''_m\left(j^1_{mk}\right), \tag{2.83}$$

which implies with Fermat's theorem and the second derivative test that all local extrema are critical points and vice versa. The second derivative test and (2.83) ensure: j^1_{mk} is a (local) maximum iff $J_m\left(j^1_{mk}\right) > 0$.

The equations (2.64) to (2.67) imply (2.68) and (2.69).

Step 10. Let $l \in \mathbb{N}$. Then j^1_{ml} and $j^1_{m,l+1}$ are extrema. Let us assume

$$\operatorname{sgn} J''_m\left(j^1_{ml}\right) = \operatorname{sgn} J''_m\left(j^1_{m,l+1}\right), \tag{2.84}$$

compare (2.83). In this case both extremum points j^1_{ml} and $j^1_{m,l+1}$ are either both local minima or both local maxima. Without loss of generality, let us consider minima. Since J_m is continuous and defined on $\mathbb{R}$, the function

$$h_l := J_m|_{[j^1_{ml}, j^1_{m,l+1}]}$$

must attain its global minimum and global maximum. Since j^1_{ml} and $j^1_{m,l+1}$ are already the only two local minima of h_l and no maximum exists by our assumption, the function h_l itself must be globally constant which contradicts (2.83). Therefore the assumption (2.84) is wrong, therefore the set $\left\{j^1_{ml}, j^1_{m,l+1}\right\}$ contains exactly one local minimum and one local maximum, and we have proven

$$\operatorname{sgn} J''_m\left(j^1_{ml}\right) = -\operatorname{sgn} J''_m\left(j^1_{m,l+1}\right) \neq 0,$$

compare (2.83). The intermediate value theorem implies the existence of a root of the function h''_l hence the sequence $\left(j^2_{mk}\right)_{k \in \mathbb{N}}$ exists as stated.

Step 11. In case $m \in \mathbb{R}_{\geq 0}$, the Bessel function of first kind and order m is given by

$$J_m(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(l+m+1)} \left(\frac{x}{2}\right)^{2l+m} \tag{2.85}$$

$$= \left(\frac{x}{2}\right)^m \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(l+m+1)} \left(\frac{x}{2}\right)^{2l}, \tag{2.86}$$

where Γ is the Gamma function. This result follows with the *Frobenius method*. For $m \in \mathbb{N}_0$ we get another representation of the function J_m as in (2.60), hence (2.70). The radius of convergence of the series of (2.86) can be calculated by the ratio test or even simpler by comparison with the power series of the exponential map. Standard arguments and (2.70) can be used to prove (2.71).

Thus all properties are proven. □

Remark 78. It is easy to verify that J_m for $m \in \mathbb{N}_0$ is an entire function. Let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Then the function

$$\begin{aligned} J_{m,\zeta}\colon\ \mathbb{R} &\to \mathbb{C} \\ h &\mapsto J_m(\zeta h) \end{aligned}$$

delivers the particular interesting functions $\Re J_{m,\zeta}$ and $\Im J_{m,\zeta}$.

The functions $J_{m,\pm 1}$ obviously do not provide any interesting new insights. From (2.70), we deduce

$$J_{m,\pm i}(h) = (\pm i)^m \sum_{l=0}^{\infty} \frac{1}{l!(l+m)!} \left(\frac{h}{2}\right)^{2l+m},$$

whose only real root for $m > 0$ obviously is at the origin. The *modified Bessel function of first kind* $I_m = i^m J_{m,i}$ is strictly increasing on $\mathbb{R}_{>0}$.

More interesting are values with $\Re\zeta\Im\zeta \neq 0$, for example $\zeta_8 = \exp\frac{\pi i}{4}$: These functions are unbounded and oscillating, compare figure 14. The real and imaginary parts are basically known as the *Kelvin functions*[49]

$$\Re J_{m,\zeta_8^3} = \mathrm{ber}_m, \quad \Im J_{m,\zeta_8^3} = \mathrm{bei}_m,$$

compare [AS72, formula 9.9.1] or [OLBC10, section 10.61].

Example 79. If we set $\theta = \pi/2, r_0 = 0$ as in example 46 we get the model of a circular membrane, compare figure 15b. The Laplacian with respect to (h, s, ϕ) of a smooth function u on the membrane then is

$$\Delta u = \left(u_{;hh} + \frac{1}{h}u_{;h}\right) + \frac{1}{t^2}u_{;ss} + \frac{1}{h^2}u_{;\phi\phi}.$$

With the Bessel function J_m, we see that for any $C, \phi_0, \tau_0 \in \mathbb{R}$, any $m, n \in \mathbb{N}_0$, any $\mu, \nu \in \mathbb{C}$ and

$$\begin{aligned} \lambda^2 &= \nu^2 - \left(\frac{\mu}{t}\right)^2 \geq 0 \qquad (2.87) \\ R(h) &= J_m(\nu h), \\ S(s) &= \exp(\mu s), \\ \Phi(\phi) &= \sin(m\phi + \phi_0), \\ T(\tau) &= \sin(c\lambda\tau + \tau_0), \end{aligned}$$

a solution of the wave equation is given by

$$u(h, s, \phi, \tau) = CR(h)S(s)\Phi(\phi)T(\tau). \qquad (2.88)$$

The appearance of the Bessel function is natural, as the Helmholtz equation transforms to the Bessel equation, compare (2.34) and (2.58). The *Bessel function of second kind* Y_m is not compatible with our problem. The equation (2.47) implies $m > 0$, since (2.71). The equations (2.48) and (2.71) imply $m > 1$. The conditions (2.49) and (2.50) are satisfied. In order to get $L^2(\Sigma_{rt})$-orthogonal solutions, we need $\mu = 2\pi\widehat{\mu}i$ for $\widehat{\mu} \in \mathbb{Z}$, note the analogy

[49] These functions are named after the British mathematical physicist William Thomson, 1st Baron Kelvin.

to $L^2\left(S^1\right)$. The condition (2.51) is connected to the values of ν and m since these determine the number of roots of the function

$$\begin{aligned} R\colon [0,L] &\to \mathbb{R} \\ h &\mapsto J_m(\nu h). \end{aligned}$$

Basically the functions $\Re R$ and $\Im R$ for all values of $\nu \in \mathbb{C}$ are interesting. In order to satisfy (2.87), we have

$$\nu^2 \geq -\frac{4\pi^2\widehat{\mu}^2}{t^2} \Leftrightarrow \nu \in \mathbb{R} \cup \left\{ i\xi \,\middle|\, \xi \geq \frac{4\pi^2\widehat{\mu}^2}{t^2} \right\}.$$

From (2.61) and remark 78 we deduce without loss of generality: $\nu \geq 0$. We set $j_{m,0} = 0$, then $j_{m,n} \leq \nu H < j_{m,n+1}$.

We use a heuristic to determine ν: We assume that the peak amplitudes of the vibrations on the mouth of the bell are (local) maxima or (local) minima, respectively, which means that νH is an extremum point of the function $J_m\colon \mathbb{R} \to \mathbb{R}$, hence

$$\begin{aligned} \nu &= \frac{j^1_{m,n+1}}{H}, \\ \lambda &= \sqrt{\left(\frac{j^1_{m,n+1}}{H}\right)^2 + \left(\frac{2\pi\widehat{\mu}}{t}\right)^2}. \end{aligned}$$

With these choices for $m, \widehat{\mu}, \nu$ defined by the modes $[m,n]$ and $\phi_0, \tau_0 \in \left\{0, {}^{\pi}\!/_{2}\right\}$, we have a $L^2(\Sigma_{rt})$-orthogonal Schauder basis of solutions of the wave equation, since $\widehat{g}$ as in (2.75) is proportional to $\sqrt{g}$, and thus can be simply adjusted to get an $L^2(\Sigma_{rt})$-orthonormal Schauder basis, compare *Fourier–Bessel series* or *Dini series*, compare [Bow58, sections 97, 98] or [Kor02, section 24].

Note that (2.67) implies $J_m\left(j^1_{mk}\right) \neq 0$ and $J''_m\left(j^1_{mk}\right) \neq 0$, thus the extremum points are exactly the critical points, which together with (2.48), form (partial) Neumann boundary conditions for R, and the local minima and local maxima have negative values or positive values, respectively.

In the most simple case $\mu = 0$, without loss of generality implying $\lambda = \nu$, which allows us to finally conclude that the overtone of the mode $[m, n]$ is exactly

$$P_{mn} \cdot \mathsf{m}_2 = \left(\operatorname{ls} j^1_{m,n+1} - \operatorname{ls} j^1_{21} \right) \cdot \mathsf{m}_2 \tag{2.89}$$

higher than the fundamental tone, using this paragraph's heuristics, see table 10b for some examples of P_{mn}.

The function (2.88) with the restrictions from above is compatible with the Chladni-patterns of figure 11. Observation 28 is also compatible with the function $u(\,\cdot\,, s, \phi, \tau)$ as in (2.88).

Note that we neither claimed, nor proved, that our Schauder basis spans the whole space of solutions of the wave equation.

Remark 80. Using different boundary conditions, but basically the same function from example 79, we get a model of a string of length t and diameter $2H$ as used on string instruments, since strings are basically of cylindric shape, figure 15b. Note that this model does not cover the oscillations of sound boxes or sound boards.

Since the string is fixed at both ends,

$$\lim_{s \searrow 0} S(s) = 0, \qquad \lim_{s \nearrow 1} S(s) = 0,$$

we get $\mu \in 2\pi i\,\mathbb{Z}$, hence without loss of generality $S(s) = \sin(\widehat{\mu}\pi s)$ with $\widehat{\mu} \in \mathbb{Z}_{\geq 0}$. The smooth function $R\colon (0, H) \to \mathbb{R}$ has no roots and $\lim_{h\to 0} R(h) \neq 0$, thus $m = 0$ and $\nu \in [0, j^0_{0,1})$, compare (2.71), hence without loss of generality $\phi_0 = \pi/2$. Note $j^0_{0,1} \approx 2.4048255576958$. This defines the sequence

$$\lambda_k = \sqrt{\nu^2 + \left(\frac{k\pi}{t}\right)^2}$$

for $k \in \mathbb{N}_0$.

The usual ansatz from physics is $\nu = 0$, thus without loss of generality for any $C \in \mathbb{R}, \tau_0 \in \left\{0, \pi/2\right\}$, and any $k \in \mathbb{N}_0$, the function

$$u(h, s, \phi, \tau) = C \sin(k\pi s) \sin\left(c\frac{k\pi}{t}\tau + \tau_0\right).$$

is a solution of the wave equation modeling a string, compare also example 67 for $d = 1$.

Example 81. It should be noted that the model of a cylindrical shell is less simple than example 79, compare figure 15c. A naive ansatz does only offer one trivial solution: If we set $\theta = 0$ and $r_0 > t = \text{const.}$ as in example 46, we get the model of a shell. The Laplacian with respect to (h, s, ϕ) of a smooth function u on the shell then is

$$\Delta u = u_{;hh} + \frac{1}{t^2}\left(u_{;ss} - \frac{t}{\varpi}u_{;s}\right) + \frac{1}{\varpi^2}u_{;\phi\phi}, \tag{2.90}$$

with $\varpi(s) = (r_0 - st) > 0$, compare (2.34). For any $m, n \in \mathbb{N}_0$ and any real numbers $C, \phi_0, \tau_0, \lambda \in \mathbb{R}, \mu, \nu \in \mathbb{C}$ and $s_{JY} \in \{0, 1\}$, the definitions

$$\begin{aligned}
\lambda^2 &= \nu^2 - \mu^2 \geq 0, && (2.91)\\
S(s) &= \begin{cases} (1 - s_{JY})J_m\Big(\nu\varpi(s)\Big) + s_{JY}Y_m\Big(\nu\varpi(s)\Big), & \nu \neq 0 \\ \delta_{m0}, & \nu = 0 \end{cases},\\
R(h) &= \exp{(\mu h)},\\
\Phi(\phi) &= \sin{(m\phi + \phi_0)},\\
T(\tau) &= \sin{(c\lambda\tau + \tau_0)},
\end{aligned}$$

where Y_m is the *Bessel function of second kind*, define the solution

$$u(h, s, \phi, \tau) = CR(h)S(s)\Phi(\phi)T(\tau) \tag{2.92}$$

of the wave equation. Similar to example 79, the radial component of the solution is covered by the Bessel equation. We note, that

$$\begin{aligned}
Y_m \colon\ &\mathbb{R}^* \to \mathbb{R}\\
&h \mapsto \lim_{\alpha \to m} \frac{J_\alpha(h)\cos(\alpha\pi) - J_{-\alpha}(h)}{\sin(\alpha\pi)}
\end{aligned} \tag{2.93}$$

is a smooth solution of the Bessel equation of order m with a singularity on 0, since $\lim_{h\to 0} Y_m(h) = \infty \in \overline{\mathbb{C}}$.

The boundary conditions (2.47) and (2.48) imply $R \equiv \text{const.}$ and $R(0) = 0$, thus $R \equiv 0$, hence the only existing solution as in the naive ansatz (2.92) is trivial: $u \equiv 0$. However, these conditions can be satisfied applying corollary 69, hence with

$$R(h) = \cos \frac{\pi \widehat{\mu} h}{H}$$

for $\widehat{\mu} \in \mathbb{N}_0$, we get an $L^2(\Sigma_{rt})$-orthogonal Schauder basis with respect to h. Thus the conditions (2.47), (2.48) and (2.51) may be satisfied using a series of u as in (2.92). For example, the functions

$$h \mapsto \frac{h^2}{H^3}(3H - 2h) \Rightarrow c_k = \begin{cases} 1, & k = 0 \\ 24\dfrac{(-1)^k - 1}{\pi^4 k^4}, & k > 0 \end{cases},$$

$$h \mapsto \frac{h^2}{H^4}(5H - 4h)(2h - H) \Rightarrow c_k = \begin{cases} \dfrac{7}{15}, & k = 0 \\ 24\dfrac{9(-1)^k + 7}{\pi^4 k^4}, & k > 0 \end{cases}$$

are simple models for $n = 0$ or $n = 1$, respectively. The conditions (2.49) and (2.50) are satisfied. To satisfy (2.91), we see

$$\nu^2 \geq -\frac{\pi^2 \widehat{\mu}^2}{H^2} \Leftrightarrow \nu \in \mathbb{R} \cup \left\{ i\xi \,\middle|\, \xi \geq \frac{\pi \widehat{\mu}}{H} \right\}.$$

From (2.61) and remark 78 we deduce without loss of generality: $\nu \geq 0$ and $\nu > 0$ for $s_{JY} = 1$.

To compare our situation with (2.66), we look at the s-component of the $L^2(\Sigma_{rt})$-scalar product with a trivial transformation

$$\int_0^1 f(\nu \varpi(s)) \varpi(s) \, \mathrm{d}s = \frac{1}{t} \int_{r_0 - t}^{r_0} f\left(\frac{j^1_{m,k}}{r_0} \xi \right) \xi \, \mathrm{d}\xi$$

of a placeholder function f with $\nu = j^1_{m,k}/r_0$. Obviously our $L^2(\Sigma_{rt})$-scalar product does only integrate over $[r_0 - t, r_0]$ and not over $[0, r_0]$ with respect to s, therefore the s-component is not orthogonal with respect to the

$L^2(\Sigma_{rt})$-scalar product. However choosing ν as above and $s_{JY} = 0$ gives us a generating set that spans a vector space of solutions by restriction of the Schauder basis of Bessel functions J_m to the given interval, compare observation 73.

Without loss of generality, these conclusions lead to

$$\lambda = \sqrt{\left(\frac{j^1_{m,k}}{r_0}\right)^2 + \frac{\pi^2 \widehat{\mu}^2}{H^2}}.$$

The thickness has an influence by the number k modeling the number of roots of S.

Example 82. For $\theta \in (0, \frac{\pi}{2})$ as in example 46, we get another bell shape which is likely more realistic than the shell of example 81. We call it conical shell, compare figure 15d. With constants $m, n \in \mathbb{N}_0$ and any real numbers $C, \phi_0, \tau_0 \in \mathbb{R}, \lambda > 0$ and $s_{JY} \in \{0, 1\}$, we apply

$$\begin{aligned}
\varpi(h, s) &= r_0 + h \sin\theta - st \cos\theta, \\
\nu(h, s) &= \lambda \varpi(h, s), \\
y(\xi) &= (1 - s_{JY}) J_m(\xi) + s_{JY} Y_m(\xi), \\
\Phi(\phi) &= \sin(m\phi + \phi_0), \\
T(\tau) &= \sin(c\lambda\tau + \tau_0), \\
u(h, s, \phi, \tau) &= C y(\nu(h, s)) \Phi(\phi) T(\tau)
\end{aligned}$$

on the Helmholtz equation

$$\begin{aligned}
0 &\stackrel{!}{=} \Delta u + \lambda^2 u \\
&= \left(u_{;hh} + \frac{\varpi_{;h}}{\varpi} u_{;h}\right) + \frac{1}{t^2}\left(u_{;ss} + \frac{\varpi_{;s}}{\varpi} u_{;s}\right) + \frac{1}{\varpi^2} u_{;\phi\phi} + \lambda^2 u \\
&= \left(u_{;hh} + \frac{1}{t^2} u_{;ss}\right) + \frac{1}{\varpi}\left(\varpi_{;h} u_{;h} + \frac{\varpi_{;s}}{t^2} u_{;s}\right) + \frac{1}{\varpi^2} u_{;\phi\phi} + \lambda^2 u
\end{aligned}$$

$$= \left(\lambda^2\left(\varpi_{;h}^2 + \frac{\varpi_{;s}^2}{t^2}\right) y'' \circ \nu + \frac{\lambda}{\varpi}\left(\varpi_{;h}^2 + \frac{\varpi_{;s}^2}{t^2}\right) y' \circ \nu\right) CT\Phi$$
$$+ \left(\lambda^2 - \left(\frac{m}{\varpi}\right)^2\right) u$$
$$= \left(\lambda^2 y'' \circ \nu + \frac{\lambda}{\varpi} y' \circ \nu + \left(\lambda^2 - \left(\frac{m}{\varpi}\right)^2\right) y \circ \nu\right) CT\Phi$$
$$= \frac{1}{\varpi^2}\left(\nu^2 y'' \circ \nu + \nu y' \circ \nu + \left(\nu^2 - m^2\right) y \circ \nu\right) CT\Phi$$
$$= \frac{0}{\varpi^2} CT\Phi,$$

hence u solves the wave equation.

The conditions (2.49) and (2.50) are satisfied. The boundary conditions (2.47) and (2.48) are not reasonable here, since our model does not contract on $h = 0$, to be more precise: In this case at $h = 0$ the boundary $S^1 \times [0, 1]$ should contract to $[0, 1]$ and Σ_{rt} should still be injective. The condition (2.51) and $L^2(\Sigma_{rt})$-orthogonality might be satisfied with proper values of s_{JY}, λ. We now deal with the s-component and h-component of the $L^2(\Sigma_{rt})$-scalar product and see with a trivial integration

$$\int_0^1 \int_0^H f'''(\lambda\varpi(h, s)) t\varpi(h, s) \, \mathrm{d}h \mathrm{d}s$$
$$= \frac{\left[\xi f'(\xi) - 2f(\xi)\right]_{\lambda\varpi(0,0)}^{\lambda\varpi(0,1)} - \left[\xi f'(\xi) - 2f(\xi)\right]_{\lambda\varpi(H,0)}^{\lambda\varpi(H,1)}}{\lambda^3 \sin\theta \cos\theta} \qquad (2.94)$$

of a smooth placeholder function f, which is hard to calculate for our case

$$f'''(\xi) = J_m(\xi) J_m\left(\tilde{\lambda}/\lambda \xi\right)$$

to choose $\lambda, \widetilde{\lambda}$ such that we gain an orthogonal basis. The term (2.94) can be expressed with a *generalized hypergeometric function* for $\widetilde{\lambda} = \lambda$, note

$$J_m(\xi) = \frac{\xi^{2m}}{4^m m!} {}_0F_1\left(; m+1; -\frac{\xi^2}{4} \right).$$

We observe that the *Robin boundary condition*[50] $\xi f'(\xi) + \varrho f(\xi) = 0$ for a constant $\varrho \in \mathbb{R}$ which is associated to Dini series might be useful on a further investigation on (2.94) to achieve orthogonality.

This model shows, that the Bessel functions are an important aspect of the oscillations on a more reasonable model of a bell, resulting in oscillations and nodes.

Example 83. We also might look at a hemispherical shell, compare figure 15e. For a general approach $0 < H < \pi/2$. In contrast to the other examples, this one is a sketch and the calculations are left to the interested reader as an exercise.[51]

We define the exterior boundary via

$$\begin{aligned} \Gamma\colon [0, H] &\to \mathbb{R}^2_{>0} \\ h &\mapsto \begin{pmatrix} 1 - \cos h \\ \sin h \end{pmatrix}. \end{aligned}$$

The unit sphere can be scaled to any radius, compare observation 64. We set const. $\equiv t \in (0,1), \varpi(s) = 1 - ts$ and calculate for $u \in W^{2,2}(\Sigma_{rt})$

$$\begin{aligned} g(h, s, \phi) &= \begin{pmatrix} \varpi^2(s) & 0 & 0 \\ 0 & t^2 & 0 \\ 0 & 0 & \varpi^2(s) \sin^2 h \end{pmatrix}, \\ \Delta u(h, s, \phi) &= \frac{u_{;hh}(h, s, \phi) + \cot(h) u_{;h}(h, s, \phi) + \sin^{-2}(h) u_{;\phi\phi}(h, s, \phi)}{\varpi^2(s)} \\ &\quad + \frac{1}{t^2} u_{;ss}(h, s, \phi) - \frac{2}{t\varpi(s)} u_{;s}(h, s, \phi). \end{aligned}$$

[50] The conditions is named after the French mathematician Victor Gustave Robin.

[51] The exercise may also include investigating the orthogonality properties with Sturm-Liouville theory as introduced above and checking the boundary conditions.

We define for $h_{PQ}, s_{jy} \in \{0,1\}, C, \phi_0, \tau_0, m, l \in \mathbb{R}$

$$\begin{aligned}
\Phi(\phi) &= \sin(m\phi + \phi_0), \\
R(h) &= (1 - h_{PQ})P_l^m(\cos h) + h_{PQ}Q_l^m(\cos h), \\
S(s) &= (1 - s_{jy})j_l(\lambda\varpi(s)) + s_{jy}y_l(\lambda\varpi(s)), \\
T(\tau) &= \sin(c\lambda\tau + \tau_0), \\
u(u, h, s, \tau) &= CR(h)S(s)\Phi(\phi)T(\tau),
\end{aligned} \tag{2.95}$$

where P_l^m is the *associated Legendre*[52] *function of first kind and degree l and order m* and Q_l^m is the *associated Legendre function of second kind and degree l and order m*, which are both solutions of

$$(1-\xi^2)\ddot{x}(\xi) - 2\xi\dot{x}(\xi) + \left(l(l+1) - \frac{m^2}{1-\xi^2}\right)x(\xi) = 0 \tag{2.96}$$

and j_l is the *spherical Bessel function of first kind and order l* and y_l is the *spherical Bessel function of second kind and order l*, which are both solutions of

$$\xi^2\ddot{x}(\xi) + 2\xi\dot{x}(\xi) + \Big(\xi^2 - l(l+1)\Big)x(\xi) = 0. \tag{2.97}$$

Applying lemma 74 and $y = \cos$ on (2.96), we get

$$\ddot{x}(h) + \cot(h)\dot{x}(h) - \frac{m^2}{\sin^2 h}x(h) = -l(l+1)x(h),$$

similarly applying lemma 74 and $y = \lambda\varpi$ on (2.97), we get

$$\frac{1}{t^2}\ddot{x}(s) - \frac{2}{t\varpi(s)}\dot{x}(s) - \frac{l(l+1)}{\varpi^2(s)}x(s) = -\lambda^2 x(s).$$

This u as in (2.95) is a solution to the wave equation. This solution is not as trivial as the other ones.

As usual $m \in \mathbb{N}$ models the meridian nodes, compare (2.49) and (2.50). Boundary condition (2.47) requires $m \leq l \in \mathbb{N}$ and ensures $P_l^m(\cos 0) = 0$, hence (2.47) and (2.48) are satisfied. Since $h \mapsto Q_l^m(\cos h)$ is singular at

[52] This function is named after the French mathematician Adrien-Marie Legendre.

0, we set $h_{PQ} = 0$. In our case, n is the number of real roots (without multiplicity) of the function

$$P_l^m(\xi) = \frac{(-1)^m}{l!2^l}(1-\xi^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}^{l+m}\xi}(\xi^2-1)^l$$

on the open interval $(0, \cos H)$ to satisfy (2.51). Using the *spherical Bessel functions of order m*

$$j_m(\xi) = \sqrt{\frac{\pi}{2\xi}} J_{m+1/2}(\xi),$$
$$y_m(\xi) = \sqrt{\frac{\pi}{2\xi}} Y_{m+1/2}(\xi)$$

in (2.97), we see that these are two linear independent solutions, using the general Bessel equation of order $m + 1/2$. In particular $j_m(j^0_{m+1/2,k}) = 0$ for all $k \in \mathbb{N}$, but $j^1_{m+1/2,k}$ are not critical points of j_m. With (2.85), we see using the *Legendre duplication formula* for the Γ-function

$$\begin{aligned} j_m(\xi) &= \sqrt{\frac{\pi}{2\xi}} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(l+m+1+1/2)} \left(\frac{\xi}{2}\right)^{2l+m+1/2} \\ &= \sum_{l=0}^{\infty} \frac{(-1)^l 4^{m+l}(l+m)!}{l!(2l+2m+1)!} \left(\frac{\xi}{2}\right)^{2l+m} \\ &= 2^m \sum_{l=0}^{\infty} \frac{(-1)^l (l+m)!}{l!(2l+2m+1)!} \xi^{2l+m}, \end{aligned}$$

which is analytic. The assumptions $S(0) \neq 0 \neq S(1)$ are plausible, implying conditions for λ. Since

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} j_m(\lambda\varpi(s)) = 0 \Leftrightarrow \lambda J'_{m+1/2}(\lambda) - 1/2 J_{m+1/2}(\lambda) = 0, \tag{2.98}$$

the s-component can be interpreted as a *Dini series* using this specific boundary condition, compare (2.81), (2.80) and [Bow58, section 98] or [Kor02, page 98].

If we had $t = 1$ and $H = \pi$ we would have a whole ball or more precisely the disc D^3 and in this case $\lambda = j^0_{m+1/2,k}$ for $k \in \mathbb{N}$, $s_{jy} = 0$ or λ as in (2.98) would give an $L^2(\Sigma_{rt})$-orthogonal Schauder basis of solutions, since

$$\begin{aligned}\int_0^\pi P_l^m(\cos h)P_k^m(\cos h)\sin h \,\mathrm{d}h &= \int_{-1}^1 P_l^m(\xi)P_k^m(\xi)\,\mathrm{d}\xi \\ &= \delta_{lk}\frac{(l+m)!}{(l+1/2)(l-m)!},\end{aligned}$$

compare [AS72, formulas 8.14.10, 8.14.13] and observation 73. This case is similar to the s-component of example 81.

We note that the two-dimensional problem of a bell with negligible thickness, is much simpler. In that case, we had for any $H \in (0, \pi/2)$

$$\Delta u(h,\phi,\tau) = u_{;hh}(h,\phi,\tau) + \cot(h)u_{;h}(h,\phi,\tau) + \frac{1}{\sin^2 h}u_{;\phi\phi}(h,\phi,\tau),$$

hence a solution would be

$$u(h,\phi,\tau) = CP_l^m(\cos h)\sin(m\phi+\phi_0)\sin(c\tau l\sqrt{1+1/l}+\tau_0),$$

with the same values for $C, c, l, m, n, \phi_0, \tau_0$, but these solutions are not $L^2(\Sigma_{rt})$-orthogonal.

Remark 84. We deduce the two-dimensional model (without thickness) for comparison. In this case the embedding is

$$\begin{aligned}\Sigma_r\colon (0,H)\times\mathbb{R} &\to \mathbb{R}^3 \\ (h,\phi) &\mapsto \begin{pmatrix}\Gamma_r^1(h)\\ \Gamma_r^2(h)\cos\phi \\ \Gamma_r^2(h)\sin\phi\end{pmatrix},\end{aligned}$$

hence the Riemannian metric is

$$g = \begin{pmatrix} \left|\dot\Gamma_r\right|^2 & 0 \\ 0 & \left(\Gamma_r^2\right)^2\end{pmatrix},$$

which can be calculated easily or deduced from (2.27). The Helmholtz equation thus is

$$\begin{aligned} 0 &= \left(\Delta + \lambda^2\right) u \\ &= \frac{u_{;hh}}{\left|\dot{\Gamma}_r\right|^2} + \left(\frac{\dot{\Gamma}_r^2}{\left|\dot{\Gamma}_r\right|^2 \Gamma_r^2} - \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left|\dot{\Gamma}_r\right|}{\left|\dot{\Gamma}_r\right|^3} \right) u_{;h} + \frac{u_{;\phi\phi}}{\left(\Gamma_r^2\right)^2} + \lambda^2 u, \end{aligned} \tag{2.99}$$

in particular for $u(h, \phi) = f(h)\sin(m\phi), m \in \mathbb{N}$ and Γ_r parameterized to arc-length

$$0 = \left(\Delta + \lambda^2\right) u = u_{;hh} + \frac{\dot{\Gamma}_r^2}{\Gamma_r^2} u_{;h} + \left(\lambda^2 - \frac{m^2}{\left(\Gamma_r^2\right)^2} \right) u$$

which is equivalent to the linear ordinary differential equation of order two

$$0 = \ddot{f} + \frac{\dot{\Gamma}_r^2}{\Gamma_r^2} \dot{f} + \left(\lambda^2 - \frac{m^2}{\left(\Gamma_r^2\right)^2} \right) f$$

which is itself equivalent to

$$0 = \Gamma_r^2 \ddot{f} + \dot{\Gamma}_r^2 \dot{f} + \left(\lambda^2 \Gamma_r^2 - \frac{m^2}{\Gamma_r^2} \right) f$$

which satisfies the Sturm-Liouville condition without an additional transformation.

Comparing the Laplacian in (2.99) with the Laplacian in example 45, we see that both Laplacians are equal except for the missing s-component. This is anything, but surprising, since the same correspondence holds for the Riemannian metrics.

Example 85. Let $L > W > 0, M = (0, W) \times (0, L) \times (0, 1) \subseteq \mathbb{E}^3$ and $g_{\mathbb{E}^3}$ the Riemannian metric of $\mathbb{E}^3$, then $(M, g_{\mathbb{E}^3}|_M)$ is a Riemannian manifold.

For ease of calculating and writing, we consider the smooth functions

$$\begin{aligned} e_{ABC} \colon M &\to \mathbb{C} \\ (x, y, z) &\mapsto \exp\left(2\pi i\left(\frac{A}{W}x + \frac{B}{L}y + Cz\right)\right) \end{aligned}$$

for $A, B, C \in \mathbb{Z}$. We know that the family of these functions

$$\mathcal{B} = \{e_{ABC} | A, B, C \in \mathbb{Z}\}$$

is closed under multiplication, multiplicative inversion, and complex conjugation and forms an orthogonal Schauder basis of $L^2(M)$. The Abelian[53] group $\mathbb{Z}^3$ is group isomorphic to $\mathcal{B}$. A real-valued basis is obtained by applying the operators $\Im, \Re$. So far, this is a slightly different point of view on example 67 for $d = 3$. We use this to look on a simple example of a tuning fork which shares some properties with a handchime, compare figure 15f.

Let $L > W > 2$, and

$$\begin{aligned} N &= \left[\left[(0, W) \times (0, 1)\right] \uplus \left[\left[(0, 1) \uplus (W - 1, W)\right] \times [1, L)\right]\right] \times (0, 1) \\ &= M \setminus \left[[1, W - 1] \times [1, L] \times [0, 1]\right]. \end{aligned}$$

The thickness of N can be varied by scaling the whole set. Let $g = g_{\mathbb{E}^3}|_N$, then (N, g) is a Riemannian submanifold of $(M, g_{\mathbb{E}^3}|_M)$, compare observation 73. The set N models a simple tuning fork.

We define the functional

$$\begin{aligned} \mathcal{I} \colon L^2(N) &\to \mathbb{C} \\ f &\mapsto \int_N *f, \end{aligned}$$

and $\mathcal{I}_{ABC} := \mathcal{I}(e_{ABC})$, hence

$$\left\langle e_{ABC}, e_{\widetilde{A}\widetilde{B}\widetilde{C}} \right\rangle_{L^2(N)} = \mathcal{I}\left(e_{ABC} \cdot \overline{e_{\widetilde{A}\widetilde{B}\widetilde{C}}}\right) = \mathcal{I}_{A-\widetilde{A}, B-\widetilde{B}, C-\widetilde{C}},$$

[53] These kind of groups are named after the Norwegian mathematician Niels Henrik Abel.

which leads to

$$\begin{aligned}\mathcal{I}_{ABC} &= \int_0^1 \int_0^W \int_0^1 e_{ABC}(x,y,z)\ \mathrm{d}x\mathrm{d}y \\ &\quad + \int_1^L \int_0^1 e_{ABC}(x,y,z)\ \mathrm{d}x + \int_{W-1}^W e_{ABC}(x,y,z)\ \mathrm{d}x\mathrm{d}y\mathrm{d}z, \\ &= -\int_{M\setminus N} *e_{ABC} \end{aligned} \tag{2.100}$$

or more precisely to

$$\mathcal{I}_{000} = 2L + W - 2 > 1, \qquad \mathcal{I}_{00C} = 0, \qquad \begin{aligned}\mathcal{I}_{0BC} &= 0,\\ \mathcal{I}_{A0C} &= 0,\end{aligned} \qquad \mathcal{I}_{ABC} = 0,$$

and

$$\begin{aligned}\mathcal{I}_{A00} &= \frac{(L-1)W}{\pi A} \sin \frac{2\pi A}{W} \\ &= 0 \Leftrightarrow \frac{2A}{W} \in \mathbb{Z}, \\ \mathcal{I}_{AB0} &= \frac{LWi}{2\pi^2 AB}\left(\exp \frac{2\pi Bi}{L} - 1\right) \sin \frac{2\pi A}{W} \\ &= 0 \Leftrightarrow \frac{2A}{W} \in \mathbb{Z} \vee \frac{B}{L} \in \mathbb{Z}, \\ \mathcal{I}_{0B0} &= \frac{L(W-2)i}{2\pi B}\left(\exp \frac{2\pi Bi}{L} - 1\right) \\ &= 0 \Leftrightarrow \frac{B}{L} \in \mathbb{Z}\end{aligned}$$

for $ABC \neq 0$, hence the restriction of the elements of $\mathcal{B}$ to N does not form an orthogonal Schauder basis of $L^2(N)$. This is not surprising. Obviously the identity (2.100) does not hold for an arbitrary function on N.

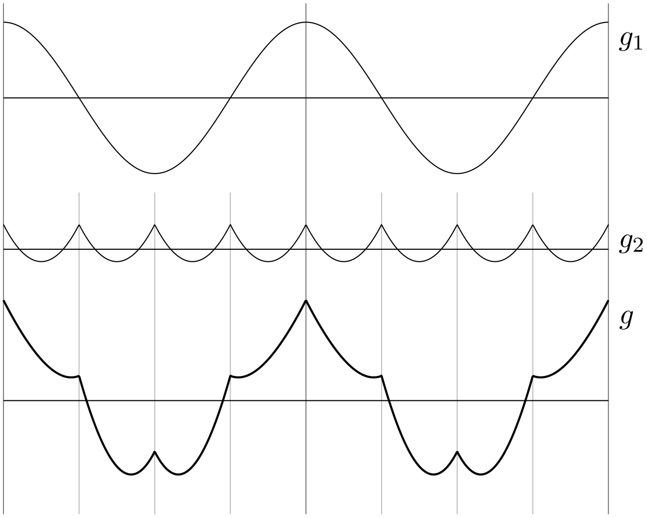
We also easily calculate

$$\Delta_g e_{ABC} = -4\pi^2 \left(\frac{A^2}{W^2} + \frac{B^2}{L^2} + C^2\right) e_{ABC}$$

which corresponds to example 67. Hence the lowest frequency of oscillation (different from zero) is c/L which is compatible to section 1.5.

So where is the difference between a tuning fork and a solid brick? The tines of the tuning fork are more flexible, hence they vibrate more easily. We most likely have different boundary conditions.

We deduce a rather surprising observation: When we damp only one tine of an oscillating tuning fork, then the other tine is damped, too, and will stop to oscillate. If we extend any e_{ABC} from N to $\mathbb{C}^3$ we get a holomorphic function. Thus the identity theorem for multivariable holomorphic functions implies this observation.



Two periods of oscillation g_1 and eight periods of oscillations g_2 are shown: The period length of g_1 is one fourth of the period length of g_2. The oscillations g_1 and g_2 generate the timbre of $g = g_1 + g_2$. Note

$$g_1 \colon \tau \mapsto \cos \frac{\pi\tau}{4}, \tag{2.101}$$

$$g_2 \colon \tau \mapsto \frac{1}{4 \sqrt[10]{10}} \sum_{k=1}^{\infty} \frac{1}{k^2} \cos{(k\pi\tau)}. \tag{2.102}$$

Figure 9: Illustration of partial tones

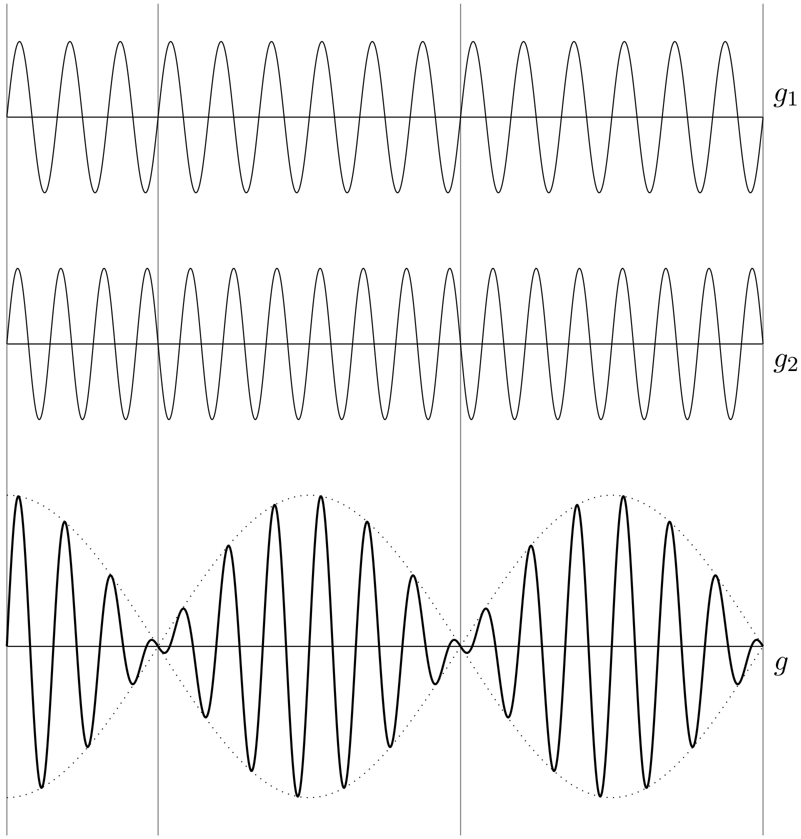
k	-14	-13	-12	-11	-10	-9	-8
$\sqrt[12]{2}^k$	0.4454	0.4719	0.500	0.5297	0.5612	0.5946	0.6300
k	-7	-6	-5	-4	-3	-2	-1
$\sqrt[12]{2}^k$	0.6674	0.7071	0.7492	0.7937	0.8409	0.8909	0.9439
k	0	1	2	3	4	5	6
$\sqrt[12]{2}^k$	1.000	1.0595	1.1225	1.1892	1.2599	1.3348	1.4142
k	7	8	9	10	11	12	13
$\sqrt[12]{2}^k$	1.4983	1.5874	1.6818	1.7818	1.8877	2.0000	2.1189

Table 8: Powers of $\sqrt[12]{2}$ and their approximation

k	1	2	3	4	5	6	7
$o_k \cdot \mathsf{m}_2{}^{-1}$	24.00	36.00	43.02	48.00	51.86	55.02	57.69
$d_k \cdot \mathsf{dB}^{-1}$	2.00	8.02	11.54	14.04	15.98	17.56	18.90

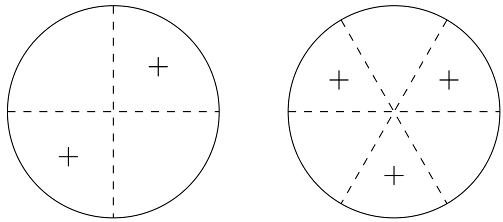
The values are approximations, except for o_1, o_2, o_4, and d_1.

Table 9: Values for example 61

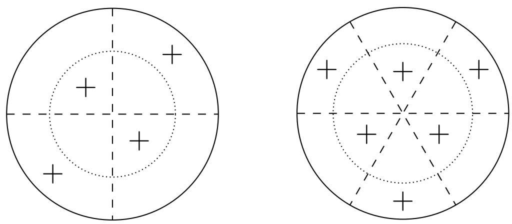


15 periods of oscillation g_1 and 17.5 periods of oscillation g_2 are shown. The oscillation $g = g_1 + g_2$ is the superposition of the former two with similar frequencies. The acoustic beat is clearly visible and highlighted by dotted lines representing envelope functions.

Figure 10: Illustration of acoustic beats

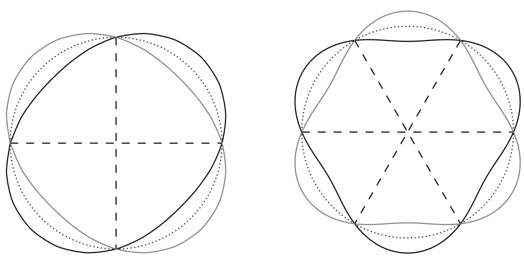


(a) mode $[2, 0]$ (b) mode $[3, 0]$



(c) mode $[2, 1]$ (d) mode $[3, 1]$

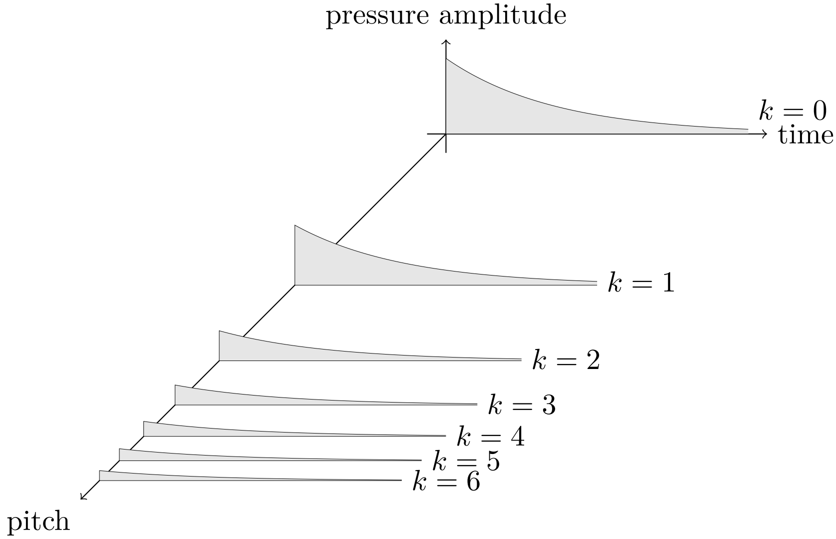
The modes of vibrations are illustrated using an orthogonal projection on a plane orthogonal to the axis of symmetry of the casting, see [Ros84, page 390]. Since the casting is fixed on the handle, there are no oscillations at the center. A mode with m *meridian nodes* and n *circular nodes* is written as $[m, n]$-mode. The meridian nodes are shown as dashed lines, while the circular nodes are shown as dotted lines. All segments with the plus sign vibrate with the same phase; the others phases are shifted.



(e) mode $[2, n]$ (f) mode $[3, n]$

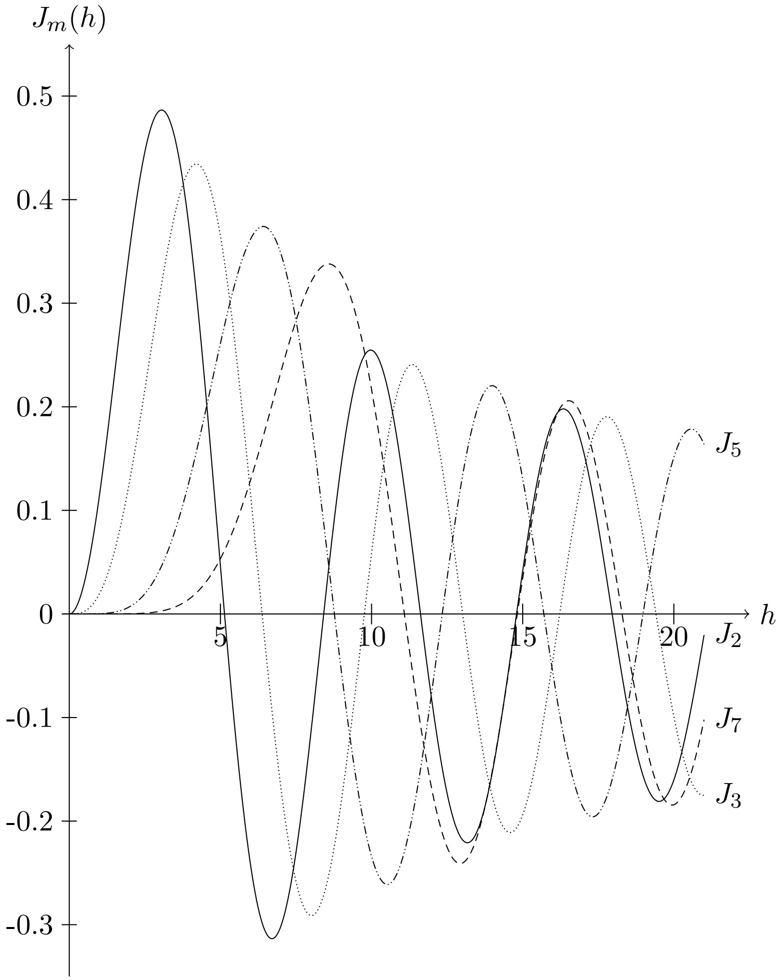
These two pictures illustrate the modes of vibration on the mouth of the casting. The dashed lines illustrate the nodes and the dotted lines show the position of a non-oscillating casting.

Figure 11: Modes of vibration of a handbell (schema)



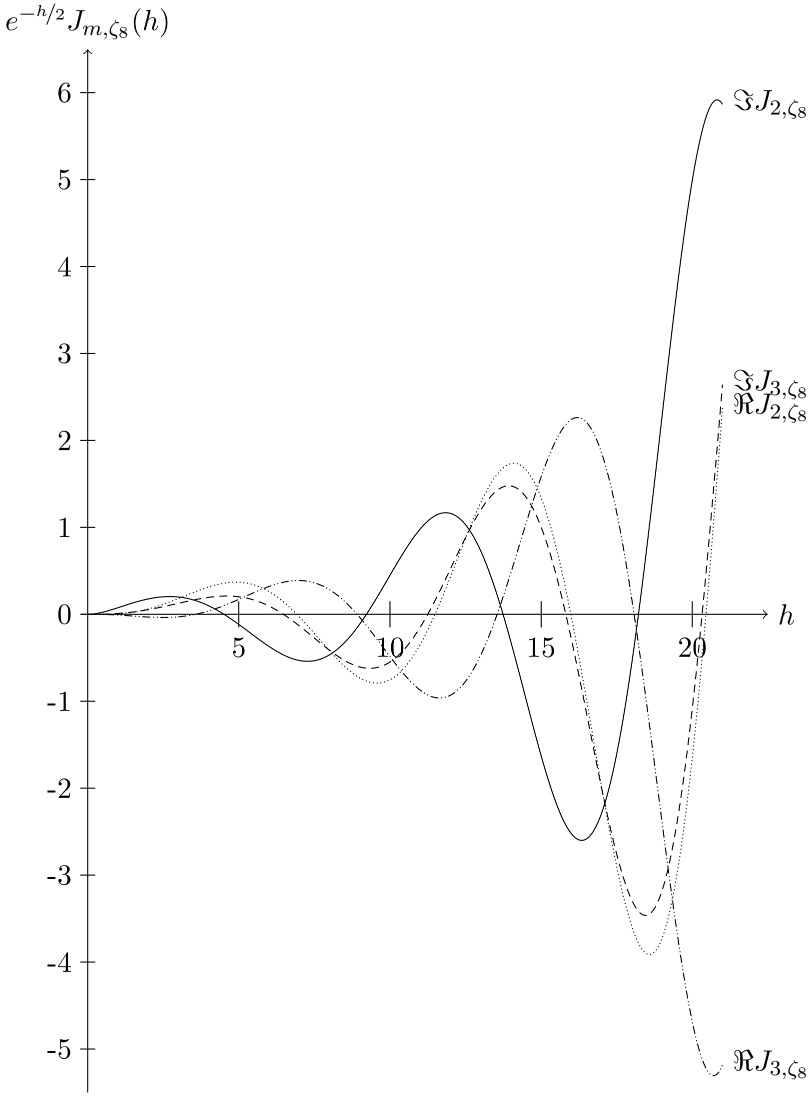
This figure illustrates example 61. The peak amplitudes of the frequencies are illustrated by envelope functions.

Figure 12: Sound decay, overtones and pitch



This figure shows the plot of J_m for some m on the interval $[0, 21]$. Note the scaled axes. Compare table 10a.

Figure 13: Plots of Bessel functions



This figure shows the plots of J_{m,ζ_8} on the interval $[0, 21]$, compare remark 78. Note the scaled axes and that all functions are unbounded although multiplied with the strictly decreasing function $h \mapsto \exp(-h/2)$.

Figure 14: Plots of Bessel functions in the complex plane

m	**k**			
	1	2	3	4
2	3.05423693	6.70613319	9.96946782	13.17037085
3	4.20118894	8.01523660	11.34592431	14.58584829
4	5.31755313	9.28239629	12.68190844	15.96410704
5	6.41561638	10.51986086	13.98718863	17.31284248
6	7.50126614	11.73493595	15.26818146	18.63744301
7	8.57783649	12.93238623	16.52936588	19.94185337

(a) Values for some extremum points j^1_{mk}

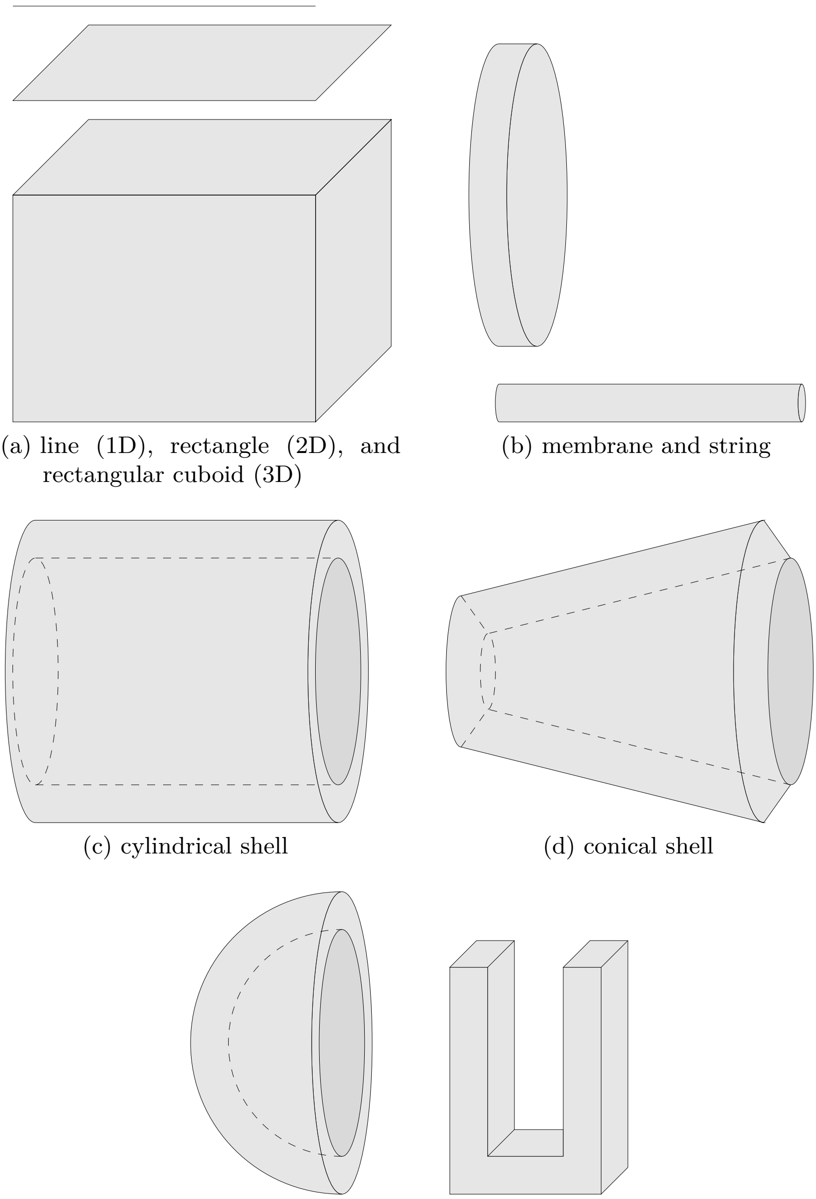
The values j_{m1} in this list correspond to a local maximum. The local maxima and minima alternate. The list does not contain all extremum points. The values in the table are approximations. Compare figure 13.

m	**n**			
	0	1	2	3
2	0	13.616	20.480	25.301
3	5.520	16.703	22.719	27.068
4	9.599	19.244	24.647	28.631
5	12.849	21.411	26.343	30.036
6	15.556	23.303	27.860	23.303
7	17.878	24.985	29.234	32.483

(b) Difference of pitch of some overtones P_{mn}

The overtone of the mode $[m, n]$ is $P_{mn} \cdot \mathsf{m}_2$ higher than the fundamental, compare (2.89). These overtones are calculated values of the model as in example 79. The list does not contain the overtones of all modes. The values in the table are approximations, except for P_{20}.

Table 10: Critical Points of J_m and Overtones of a membrane



(a) line (1D), rectangle (2D), and rectangular cuboid (3D)

(b) membrane and string

(c) cylindrical shell

(d) conical shell

(e) hemispherical shell

(f) tuning fork

Figure 15: Visualization of some shapes

Pagina vacat.

Musik wird oft nicht schön gefunden,
Weil sie stets mit Geräusch verbunden.

from Der Maulwurf
WILHELM BUSCH

3. Compositions

As seen in section 1.4 compositions have an important role in handbell music since the use of an instrument in the sense of section 1.1 is more dominant than the use as a practicing tool for change ringing.

3.1. Musical compositions

Handbell compositions are not limited to a specific kind of western music. Non-western music is not treated in this document. There may be some difficulties with certain non-Western music on handbells since handbells are pitched in the chromatic scale.

There is a large number of compositions for handbells exclusively, and some composers who dedicated themselves to handbells are well known among the community.[54]

The spectrum of arrangements contains traditional music and music of baroque, classical and romantic eras. Arrangements are usually restricted to single movements only. There also are some pieces of 20$^{\text{th}}$ century music

[54] M. Jedamzik does not list them here because forgetting some is quite embarrassing.

or new music, respectively. Arrangements of film scores or musicals are also popular for handbells.

Handbells can be played with other instruments as well. For example, some compositions include a voice for violin, others include voices for organ or voices for a brass ensemble. Percussion instruments are used, as well, for some pieces.

The one-act opera *Noye's Fludde* by Edward Benjamin Britten features a handbell ensemble.

3.2. Handbell used chart

Every sheet of handbell music should begin with the *handbell used chart*. In this chart every used handbell and used handchime is listed. Optional handbells and optional handchimes are listed as well. The number of bells needed may give a first hint for the number of needed ringers.

3.3. Difficulty levels

This whole subsection is based on [HMA10, pages 24-26]. Opera for handbells are classified depending on their difficulty on a *grading system*, which is a discrete, bounded and non-linear scale. Based on this scale, ensembles are also evaluated, see also section 4.2.

> "As handbell and handchime repertoire and techniques have increased in number and complexity, the need for a method of assigning difficulty levels has become apparent. Having music available with an assigned difficulty level will:
>
> - select repertoire best suited for their choirs,
> - help directors to select literature that requires specific skills and techniques,
> - assist teachers in creating a curriculum,
> - provide a framework for educational assessment,
> - serve as a motivational tool that encourages choirs to improve their skills

- help publishers select new releases for a balanced catalogue.

The following system should be used only as a guide. Tempo, number of ringers, handbell assignments, et cetera will have a dramatic effect on the difficulty of any music selected.

Comments for Directors, Publishers, and Editors

(1) Key changes and accidentals are handbell and handchime changes.

(2) Tempo is very important in assigning level of difficulty.

(3) Handchimes should be considered as a special category. However, when used with handbells within the same piece, a handchime should be considered a 'handbell' change.

(4) When a piece contains a six-measure (or less) phrase of technical difficulty above the specific level assigned, the piece should not be raised to the next level of difficulty. That phrase should be treated as a 'special practice' spot for learning.

(5) Shelley, four-in-hand, grace notes, and sharing of handbells are directors' decisions based on the size of the group, number of handbells, and dexterity of the ringers.

(6) Difficulty levels are assigned for 'traditional size' handbell choirs, i, e., 11 to 13 ringers.

(7) Each difficulty level is described by eight criteria. They should be used to determine the level of the work before selection.

(8) On multiple octave publications different levels may be assigned to specific octave designations. Example: a 3 - 5 octave publication may have the following designations: 3 octave – L3, and 4 - 5 octave – L4.

(9) A plus or minus may be added to any level designation when appropriate," [HMA10, pages 24-26].

The definition of the levels is given in table 11, which is based on [HMA10, pages 24-26]. The corresponding examples of figure 16 are also from [HMA10, pages 24-26]. All levels as in table 11 are cumulative: Each level contains the lower level, albeit the declared level hints at overall difficulty.

3.4. Bass, battery and treble handbells

A handbell set is typically divided roughly into three groups by tone frequency *treble*, *battery* and *bass* (descending). A clear definition is difficult or rather useless: The battery of a large handbell ensemble with a bell range of about seven octaves might coincide with the bell range of a small bell ensemble with a bell range of two octaves. It seems to be more useful to base this definition on the bell range and on individual pieces, in particular the function a bell has within a piece. Thus every ringer has to deal with conductor's instructions like "I need more treble." Sometimes the bells are classified via the clef.

Usually each of these categories is connected with different musical aspects and different difficulties. The lighter the bells the more difficult the rhythms are. Whereas multiple bell-in-hand techniques are connected to (treble) bells like C_8, a (bass) bell made of bronze like C_2 can be considered as a two-handed bell at least if forte fortissimo is required.[55] Many ringers develop preferences for one of these categories and some even for particular bells.

3.5. Assignment

The director knows his or her ringers and assigns the bells and handchimes. An intense view on assignment is given by [AHST95]. Other interesting sources are [Kin00, *retrieved* June 5, 2004] or [Sue07, 18 *Assignments*]. The *handbell used chart* gives first hints on assignment. Handchimes are usually treated similarly to handbells.

[55] The (currently) largest bronze handbell of the Handglockenchor Wiedensahl is a F_2 with an weight of $7.4 \cdot \mathsf{kg} \approx (16 + {}^5\!/_{16}) \cdot \mathsf{lb_m}$.

3.5.1. Borrowed, lent and shared bells

Since the difference of these terms may cause some confusion, the little dictionary may be helpful:

Shared bell. A shared bell is assigned to usually two ringers. That means the ringer who can ring the bell has to ring it. A synonym is *common bell*.

Borrowed bell. The bell is assigned to another ringer and is only rung occasionally by you.

Lent bell. This bell is assigned to you. On rare occasions another ringer rings this bell because it is impossible for you to do so.

Obviously borrowing and lending one specific handbell is the same concept, but this definition emphasizes views of different ringers.

Usually the ringers manage borrowing and lending bells by themselves: The conductor does not have to be involved. This is different from a shared bell: The intention of sharing a bell is a concept of some assignment strategies and is intended by the assigning one. Borrowed, lent and shared bells should be marked in sheet music with a pencil.

3.5.2. A classical assignment example

The *classic way to assign handbells* is giving each ringer the same bells independently from the piece. One slightly altered example of [Kin00, *retrieved* June 5, 2004] can be seen in table 12. The classification of the numbers 0 to 11 is more or less standard, but the letters are used differently here to give a coherent system.[56]

A handbell ensemble with less than seven octaves of handbells simply omits the bells they cannot ring. This is only one example and can be considered as a basis for piece-specific assignment.

Additional handchimes are treated as bells. Another variation is splitting the ensemble in a handbell choir and one handchime choir.

[56] To give an even more systematic approach, the numbers of the positions can be shifted so that ringer 0 has bell G_5, compare section 1.9. But this would cause an incompatibility to almost all used systems.

3.5.3. Assignment notes

If ringers do not have fixed bells for a whole concert or not all bells are used for every piece, it is a good idea to write a list for every ringer. If a ringer wants to note some further information for every piece, she or he can write these notes on the assignment note, see figure 17 for an example. A ringer cannot expect the conductor to write assignment notes with other information than pieces and corresponding assignments.[57] Writing a good assignment note may require some study of the particular sheet music which may have some positive side effects because a ringer usually gets a better understanding of a particular piece. Below are some remarks listed:

(a) The order of pieces should coincide with the order in concert.

(b) The symbol $\mathcal{M}$ indicates that mallets are required; one may use the symbol **+** instead, see table 22. One can also specify and use $\mathcal{M}_x$, where x describes the mallet, for example the color of its head: $\mathcal{M}_{xy}^{21}$ for two mallets of color x and one of color y.

(c) Handchimes are usually written with lower cases. Compare table 2 for a overview of fonts used to distinguish different sets of instruments.

(d) Notes wrapped in brackets are affected by ringer change: Parentheses $(\,\cdot\,)$ for borrowed bells, square brackets $[\,\cdot\,]$ for lent bells and curly brackets $\{\,\cdot\,\}$ for shared bells, see section 3.5.1.

(e) Use of enharmonic equivalences based on key changes is indicated similar to "$\mathsf{F}_2^\sharp$=$\mathsf{G}_2^\flat$".

(f) Listing the bells as a starting setup allows putting some further information compactly.

(g) Underlining notes can be used to indicate the first instruments to pick up from table.

[57] The principle of giving every ringer a list was introduced to M. Jedamzik by Thomas Eickhoff as this is a common procedure within the Handglockenchor Wiedensahl. The system of additional information as shown here is developed by M. Jedamzik.

(h) The notes column is usually left blank if the assignment note is given by the conductor. Every ringer can make his or her own notes. The conductor may only publish a list, so each ringer has to compile her or his own assignment note.

(i) One can also mark some bells on the assignment note with colors, especially when the bells are marked with the same colors in the sheet music.

3.5.4. Assignment-exceptions and bell-hogging

There may be exceptions to the handbell assignment given by the director: If it is not possible for a ringer to ring a bell,[58] on rare occasions his or her neighbor should ring the bell instead. These situations usually show up in rehearsals. Sometimes these occasions only show up in dress rehearsal since local features may differ. In most situations the conductor must not be involved if both ringers agree to swap on a few occasions. Some assignment principles also include bell swapping as part of a concept.

Some ringers may be be addicted to take over other ringers' bells in case they might have some problems ringing all of their bells. During a concert one ringer named Rupert may suddenly notice that his neighbor Robert cannot ring a bell, so Rupert decides to ring this bell for Robert instead. Note that Rupert may neglect his own duties and may irritates Robert or the conductor. This technique is called *bell-hogging*. Ringers who tend to use bell-hogging are called *bell hogs*, see [Sue02, *Handbells. Bass Ringer's Notebook. Smarts*, *retrieved* November 8, 2012], [AHST95, pages 7, 9, 25, 26] and appendix C.2.1.

3.5.5. One note on rhythms

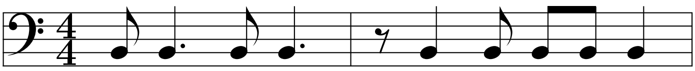
Instead of discussing whether this aspect is too trivial to be in this document, we go right to the point.

The rhythms itself of figure 18 are quite simple. What makes ringing C_3 difficult is the fact that the two prominent offbeat chords in the second measure may mislead to a wrong rhythm.

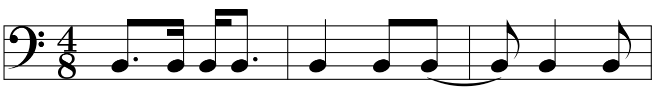
[58] One example is that bass ringers usually have at most two hands and four-in-hand is practically impossible on lower sixth or seventh octaves.

Two different strategies to deal with this are: First: Ignore the chords and just focus on $\mathsf{C_3}$. Second: See the music as a whole which may also include phantom-ringing which means also ringing the chords imaginarily or with a small subtle movement of a finger.

At Handglockenchor Wiedensahl some ringers can only ring well with one of theses strategies. This example may indicates that there are different ways of thinking musically.



(a) rhythmic example for level 3



(b) rhythmic example for level 5

Figure 16: Rhythmic examples for the difficulty level system

Lv.	meters	notes or rest values	rhythmic elements	techniques
1	$^4/_4$, C (common time), $^3/_4$ or $^2/_4$	whole, dotted-half, quarter	no subdivision of beats, simple use of ties	ring, shoulder damp, Sk, TD, echo, martellato, Sw, RT – all with adequate preparation time
2	$^2/_2$, cut time, $^3/_2$, and simple mixed meters of $^2/_4$, $^3/_4$ or $^4/_4$	eighths, the dotted-quarter followed by an eighth, simple combinations of eighths and quarters	syncopation – simple patterns such as eighth-quarter-eighth, anacrusis – pick-up notes or upbeats and their effect on the final measure	table damp, Pl, martellato-lift, malleting, and any combination of two different techniques with adequate preparation time
3	$^6/_8$, $^3/_4$ (in one pulse per measure), $^3/_8$, $^9/_8$, $^{12}/_8$ or $^6/_4$	sixteenth, dotted-eighth and sixteenth note patterns, triplet	syncopation such as seen in figure 16	Brush Damp
4	mixed of $^6/_8$ and $^3/_4$, $^5/_4$	all of previous at faster tempo, triplet over two beats	syncopation – more complex, using sixteenth-notes and ties	Brush Damp
5	irregular meters	dotted rhythms in compound meters at fast tempi, duples against triples	syncopation – more complex, mixed patterns, see figure 16 for an example	ring-hook-damp sequences, handbell passes at moderate tempi
6	unlimited	more than four eighth or sixteenth-notes to a pulse (such as five, six, seven, etc.), thirty-second notes	complex rhythms at any tempo	all, any tempo

The contents of this table are taken from [HMA10].

Table 11: Definition of difficulty levels

key changes	articulation	dynamic levels	tempo	Lv.
none (no accidentals)	see column *techniques*	all from pp to ff in homophonic style (all ringing at the same level) with limited use of crescendo or diminuendo	slow to moderate	1
limited number of changes per ringer with adequate preparation time	see column *techniques*	crescendo and diminuendo, polyphonic style with simple dynamic contrasts (such as two voices having different dynamic levels)	slow to moderate	2
extensive number of changes per ringer	combinations of techniques in eighth-note patterns at moderate tempi	subito piano or subito forte without rest, more complex polyphony with more than two independent voices, more rapid shifts of dynamic levels	more changes of tempo within the work	3
extensive number of changes per ringer	combinations of techniques in eighth-note patterns at moderate tempi	subito piano or subito forte without rest, more complex polyphony with more than two independent voices, more rapid shifts of dynamic levels	more changes of tempo within the work	4
unlimited	any combination at faster tempi	rapid shifts between levels with no preparation, more frequent use of crescendo and decrescendo	more changes of tempo within a work including abrupt shifts	5
unlimited	unlimited combinations at any tempo	no limits on shifts (sudden or gradual) or accents	only those imposed by the nature of the instrument, complex changes within a work	6

position	**Assigned handbells**					
	2nd oct.	**3rd oct.**	**4th oct.**	**5th oct.**	**6th oct.**	**7th oct.**
F						C_2 D_2
E						$E^\flat_2$ F_2
D					G_2 $G^\sharp_2$	$F^\sharp_2$
C					A_2 B_2	
B				C_3 D_3		
A			G_3	E_3 F_3		
0			$G^\sharp_3$ A_3 B_3			
1		C_4 D_4				
2		E_4 F_4				
3	G_4 A_4					
4	B_4 C_5					
5	D_5 E_5					
6	F_5 G_5					
7	A_5 B_5					
8	C_6 D_6					
9	E_6 F_6		E_7 F_7		E_8 F_8	
10	G_6	A_6	G_7	A_7	G_8	A_8
11		B_6 C_7		B_7 C_8		B_8 C_9

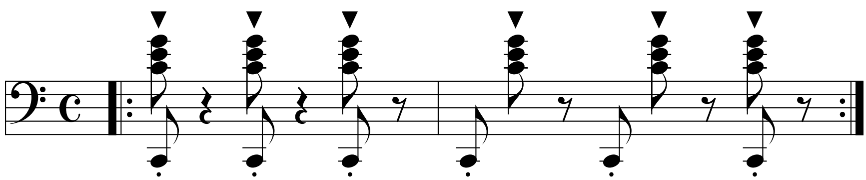
The written notes span an interval, for example position 1 has bells C_4,$C^\flat_4$, D_4. The other handbells not listed are assigned by the ringers themselves, including sharing or borrowing or lending respectively, see section 3.5.1. Position A may undertake handbells from position 0 if the ensemble does not ring with five octaves.

Table 12: A classic assignment strategy

Michael J. *Riemannian Geometry Congress*

Piece	Bells/Setup	Notes
Farandole	$\mathcal{M}_g^2$, D_2, $\mathsf{F}_2^\sharp$, A_2, B_2, $\underline{(\mathsf{E}_4^\flat)}$	bar 7: cede $\mathsf{E}_4^\flat$ to Sven and change position
Koszul formula	$\mathcal{M}_g^2$, $\mathsf{A}_2^\sharp$=$\underline{\mathsf{B}_2^\flat}$, $\underline{\mathsf{C}_2}$, A_2, B_2	
	break	
Toccata Allegro	F_2, $\mathcal{M}_g^2$, $\underline{\mathsf{G}_2}$, $\mathsf{A}_2^\flat$, $\mathsf{A}_2^\sharp$=$\mathsf{B}_2^\flat$, $\underline{[\mathsf{A}_2]}$, $\{\mathsf{B}_2\}$	
Jesus bleibet meine Freude, BWV 147.10	G_2, A_2, $\underline{\mathsf{B}_2}$	
Escape Velocity	F_2, $\underline{\mathsf{G}_2}$, $\mathcal{M}_g^1$, $\mathsf{G}_2^\sharp$=$\mathsf{A}_2^\flat$, $\mathcal{M}_g^1$, A_2, $\underline{\mathsf{B}_2^\flat}$, B_2	
Exultate	$\underline{\mathcal{M}_g^2}$, F_2, G_2, $\mathsf{A}_2^\flat$, $\mathsf{B}_2^\flat$, H_2=$\mathsf{C}_3^\flat$	
With a Joyful Heart	tacet	
You are My All in All	F_2, G_2	
	break	
Dona nobis pacem	tacet	handbell quartet
Reverberations	$\mathcal{M}_g^2$, $\underline{\mathsf{A}_2}$, $\underline{\mathsf{B}_2^\flat}$, G_2, $\mathsf{G}_2^\sharp$	
Passacaglia, BWV 582.1	$\underline{\mathsf{B}_2}$, G_2, $\underline{\mathsf{F}_2}$, C_2	pattern: B_2, F_2, G_2, C_2
Resonances and Alleluias	$\mathcal{M}_g^2$, $\underline{\mathsf{C}_2}$, G_2, $\mathsf{B}_2^\flat$, $\mathsf{A}_2^\flat$	
	encore:	
Souvenir de Cirque Renz	$\underline{\mathcal{M}_g^2}$, B_2, G_2, A_2	

Figure 17: Example for an assignment note



This example is used in section 3.5.5.

Figure 18: A rhythmic example

Pagina vacat.

To ring or not to ring, when is the question:
Whether 'tis nobler in the mind to suffer
The notes and rests of outrageous composers,
Or to take chimes against a sea of troubles,
And by damping, end them? To ring: to tacet!

from Belllet
WILLIAM SHAKEABELL

4. Handbell ensembles

A handbell ensemble consists of several handbell ringers, one director, instruments and equipment.

4.1. Musicians

Making music in an ensemble requires discipline. In a handbell ensemble there are typically two roles of musicians: One conductor and several ringers.

4.1.1. Director

We call the highest authority inside the handbell ensemble *director*. A director usually is the music director, the general manager, teacher and conductor of the handbell ensemble. It is possible to assign these responsibilities to several persons, however.

4.1.2. Conductor

The conductor is worthless if ringers do not pay enough attention to him or her. Since a handbell ensemble is one single instrument played by a dozen individuals with independent brains rather than an ensemble consisting of several instruments, all ringers have to be guided and to be merged into one well-working instrument, turning sheets and individual skills into an exceptional experience of art. If ringers and conductor interact well, the conductor is an invaluable member of the team. Hence treating a conductor with respect is a good idea. The conductor usually does not contribute to the sound experience. The conductor, however, may ring some handbells in a contingency for example one otherwise irreplaceable ringer is missing due to short-term illness, compare page 5. A typical conductor is usually more versatile in theory of music than an average ringer of the same ensemble.

Usually a conductor uses different techniques to guide a handbell ensemble which may be a challenging duty. There are several books dealing with conducting a handbell ensemble. The signs given by ensemble's conductor must be understood by every ringer. As mentioned above, every ringer has to keep an eye on the conductor especially when she or he cues after a fermata or rest.

The conductor may also instruct ringers to perform differently than what is written in sheets. If there are questions on articulation or other different matters, a ringer should feel free to ask the conductor at rehearsal.

Usually a conductor of a handbell ensemble is also its director.

Some standard resources for conductors are [All82, All85, All92], [BV88], [Ive95], [AHST95], [Tho96], [BM99], [Par06], [Sue07], [Sha11], [McC11], [Ber12], [McC16], [HMA16], the relevant manuals, and various publications by the Handbell Musicians of America, some sections of the notably older books [Fle24] and [Wat59], and probably [Jed17]. The books [Fra94] and [Hop05] cover aspects of church services.

4.1.3. Handbell ringer

As mentioned above, a handbell ensemble consists of individuals with individual strengths. Some ringers are likely more skilled than others and some have more or less balanced skills. Rehearsals and experiences may increases individual skills. Since the ensemble works as a team, arrogance or mocking usually does not contribute and should be avoided outside the musical world, too. Advanced ringers may give some hints and tips in adequate situations.

Every ringer has to check his or her handbells for damage or other problems and has to report them if the ringer cannot remedy the problem.

4.2. HMA ensemble levels

An ensemble can be rated by its level of proficiency, compare [Han11, *Fundamental Tone. Volume XX, No. 2 Winter 2006. page 14*, *retrieved* March 31, 2013]. These three levels are based on the classification as in section 3.3:

Tin. Ensembles ringing mostly level 1 to level 2^+ music.

Copper. Ensembles ringing mostly level 2^+ to level 3^+ music.

Bronze. Ensembles ringing mostly level 3^+ to level 6 music.

4.3. Clinicians and workshops

A *handbell clinician* is a person who serves as a teacher or lecturer in workshops dealing with handbell topics. The title "handbell clinician" is not a protected professional title. There is no certificate or academic degree from an independent academy. The Handbell Musicians of America developed the program "Handbell Musician Certification", see [HMA14, *Music & Resources. Handbell Musician Certification*, *retrieved* July 16, 2013].

Typical workshop topics are

(a) general information
  - (i) where to start
  - (ii) handbell history
  - (iii) physics

(b) care
  - (i) polishing
  - (ii) small repairs and interchanging small parts
  - (iii) handbell care
  - (iv) handchime care

(c) ringing and damping techniques
  - (i) table layout
  - (ii) weaving
  - (iii) bell changing
  - (iv) healthy ringing
  - (v) sight-reading

  - (vi) bass techniques
  - (vii) battery techniques
  - (viii) treble techniques
  - (ix) malleting and percussion
  - (x) multiple handbells in hand techniques
  - (xi) solo ringing
  - (xii) handchimes
  - (xiii) change ringing
  - (xiv) difficult rhythms
- (d) preparing sheet music
  - (i) color-coding
  - (ii) copyright issues
- (e) musical direction
  - (i) music theory
  - (ii) score study
  - (iii) conducting
  - (iv) starting with a new group
  - (v) assignment
- (f) performance
  - (i) visual elegance[59]
  - (ii) basics for conférenciers
- (g) introduction to composing[60]
  - (i) use of handchimes or other bells along with handbells
  - (ii) effects of specific ringing techniques
  - (iii) transcribing music
  - (iv) arranging music
  - (v) publishing music
  - (vi) software
- (h) management
  - (i) organizing a handbell festival
  - (ii) handbells in worship and liturgy
  - (iii) fund raising
  - (iv) experience and knowledge management
  - (v) motivation
  - (vi) gaining new ringers
- (i) recording
  - (i) organizing recording sessions

[59] The article [Hog13, *retrieved* February 9, 2014] dealing with the paper [Pur13, *retrieved* February 9, 2014] may be transferred to handbell ringing, showing the importance of the overall visual appearance. This article may also illustrate an opportunity a handbell ensemble has, because the motions of the handbells are unique and likely not seen on any other kind of musical instrument: The audience can "see" the music.

[60] See the article [Yor14] or its republishing [Kir11, *Handbell Articles (Blog). Composing for handbells by David York from March 21, 2015*, *retrieved* April 28, 2015] written by David York for an introduction. An excerpt is printed in **??**.

   (ii) audio recording
   (iii) video recording
   (iv) publish recordings

(j) education
   (i) the choice of the *right* instruments
   (ii) benefits in musical education
   (iii) setting up schedules
   (iv) special populations

(k) recent developments
   (i) new compositions and arrangements
   (ii) testing new products
   (iii) seeing or listening to new publications
   (iv) proper use of new technology (including new media)

(l) miscellaneous[61]

Each category may include discussions, hands-on courses or lectures. In workshops ringers can see and use instruments made by different manufacturers.

A workshop usually closes with a concert which often contains *massed ringing*, that is several handbell choirs ringing together the same piece at the same time, often conducted by a handbell clinician. Not all members have to attend a workshop: The director (for example) can attend a workshop and teach his ringers afterword.

4.4. About clothing

Musical ensembles are observed visually; therefore, every member of a handbell ensemble has to be dressed well and in most cases uniformly. The dress code is specified by the director, minister of music or a similar position. Sometimes ringers have some leeway, for example all ringers have to cover their torsos with a black garment. Some choose dress shirts while others choose sweaters. The color of gloves is usually preassigned, for example a ringer has to wear black gloves but can choose their gloves including glove material.

[61] Especially cultural aspects on multinational or multicultural symposium.

4.4.1. On shirts

Because handbells and handchimes are usually damped on a (not naked) human torso, zippers and buttons and similar non-plushy objects may generate unpleasant noise when touched by a handbell and should therefore be omitted. They may also leave scratches on the instruments. Concealed/covered placket shirts are recommended, sometimes this feature is also called *fly front*.

Bass ringers may use the *hug damp* by using one arm to damp large bells, implying a need for sleeves to protect the handbell.[62] For a uniform look, all ringers have to wear long sleeves. Besides, some directors prefer long sleeves for their more formal look.

4.4.2. About gloves

A human's skin secretes an oily or waxy matter and sweat. These fluids are necessary for the skin, but tarnish bronze and may damage handbells. There is an ever-lasting debate on the question "Gloves: To wear or not to wear?" among handbell ensembles. The aspects on the debate are:

(a) Gloves have an influence on the visual appearance.

(b) Gloves reduce the tactile perception.

(c) Gloves do not block sweat sufficiently.

(d) Gloves do block an amount of sweat, which makes cleaning the bells after each use easier.

(e) The care and maintenance instructions of section 1.11.1 tell you to clean your handbells after every use anyway.

(f) Gloves protect the bells from jewelry like wedding rings.

(g) Gloves protect the hands from wear and tear like blisters. The padding is very welcomed by many bass ringers.

[62] Side note: A bass ringer can also show his love to his instrument by using the hug damp.

A director has to make a decision based on the arguments listed (and perhaps some missing arguments) above. The arguments as listed above also depend on the specific gloves used.

Typical materials for gloves are cotton, leather and synthetic fabrics like artificial leather and have pads on specific places. Commonly used gloves are cotton riding gloves. Other types are driving gloves, biking gloves, golf gloves, sailing gloves or weightlifting gloves. LARRY J. SUE has written the two texts[63] [Sue02, *Handbells. Bass Ringer's Notebook. Gloves*, *retrieved* November 8, 2012] and [Sue07, 1.1 *Gloves*] on this subject.

Since Malmark in [Mal05, page 1] recommends wearing gloves and Schulmerich includes a set of gloves in the *Enhanced Package* of handbells, wearing the right gloves does not seem to be a bad decision. To make turning pages easier even with gloves, see section 5.1.2. For further reading the article [Kir11, *Handbell Articles (Blog). Gloves from September 21, 2011*, *retrieved* March 4, 2013] and the text [Ber12, pages 164-170] are suggested. Based on a small research on the list [HMA14, *Events & Networking. Handbell Industry Council. Buyers Guide. Professional Groups*, *retrieved* April 20, 2013] and adding some other ensembles, the following ensembles or soloists wear gloves while ringing:[64]

(1) *American Church in Paris Bronze Ringers* (Paris, FR-75)

(2) *Agape Ringers* (Carol Stream, US-IL)

(3) *Atlanta Concert Ringers* (Atlanta, US-GA)

(4) *Andante*, 「アンダンテ」 (Setagaya, JP-13)

(5) *Arsis Handbell Ensemble* (Tallin, EE-37)

(6) *Back Bay Ringers* (Boston, US-MA)

(7) *Beacon Hill Ringers* (Boston, US-MA)[65]

(8) *Bells of the Lakes* (Minneapolis, US-MN)

(9) *Bells of the Rockies* (Broomfield, US-CO)

(10) *Bells of the Sound* (Seattle, US-WA)

[63] The web page seems to be a sketch for his book [Sue07]. The web pages' text on gloves differs notably from [Sue07, 1.1 *Gloves*].

[64] Some of the ringers wear fingerless gloves.

[65] Known for being one of the oldest ensembles of the US, founded 1923, compare [Bak87].

(11) *Campanelli* (Tallin, EE-37)

(12) *Campanile* (Los Angeles, US-CA, defunct)

(13) *Capital City Ringers* (Lansing, US-MI)

(14) *Cast of Bronze* (Dallas, US-TX)

(15) *Chicago Bronze* (Elk Grove Village, US-IL)

(16) *Arsis Handbell Ensemble* (Tallin, EE-37)

(17) *Jörn Diedrichsen* (Altdorf, DE-BY)

(18) *Embellish Handbell Ensemble* (Grand Rapids, US-MI)

(19) *emBellishments Handbell Ensemble* (Hong Kong, HK)

(20) *Forté Handbell Quartet* (Colorado Springs, US-CO)

(21) *Glee Handbell Choir of Meiji Gakuin Higashimurayama High School*, 「明治 学院 グリー ハンドベル クワイア」 (Tokyo, JP-13)

(22) *Grace GosBells* (Manassas, US-VA)

(23) *Hakuoh University Handbell Choir*, 「白鴎 大学 ハンドベル クワイア」 (Oyama, JP-09)

(24) *Handglockenchor Gotha* (Gotha, DE-TH)

(25) *Handglockenchor Wiedensahl* (Wiedensahl, DE-TH)

(26) *Houston Bronze Ensemble* (Houston, US-TX)

(27) *Jubilate Ringers of Our Father Lutheran Church* (Colorado Springs, US-CO)

(28) *Käsikellokuoro Dolce* (Kuopio, FI-15)

(29) *Kinjo Gakuin University Handbell Choir*, 「金城 学院 大学 ハンドベル クワイア」 (Nagoya, JP-23)

(30) *Nancy Kirkner* (Seattle, US-WA)

(31) *Kiriku Handbell Ensemble*, 「きりく ハンドベル アンサンブル」 (Tokyo, JP-13)

(32) *Kevin Mazimas Ko* (Hong Kong, HK)

(33) *Low Ding Zone* (Palo Alto, US-CA, defunct)[66]

[66]Known for being the first bass-only handbell ensemble, founded by LARRY J. SUE.

(34) *Danny Lyons* (Pensacola, US-FL)

(35) *Elizabeth Mays* (Brea, US-CA)

(36) *Milwaukee Handbell Ensemble* (Glendale, US-WI)

(37) *Ministry of Bellz* (Singapore, SG)

(38) *National Honors Handbell Ensemble* (various, US)

(39) *Pacific Bells Handbell Ensemble* (Greater Los Angeles Area, US-CA et al.)

(40) *Palmetto Bronze* (Summerville, US-SC)

(41) *Parrish Bells* (Manassas, US-VA)

(42) *Pikes Peak Ringers* (Colorado Springs, US-CO)

(43) *Purdue Bells* (West Lafayette, US-IN)

(44) *Prime Handbell Team*, 「プライム」 (Nagoya, JP-23)

(45) *Megan Reishus* (Colorado Springs, US-CO)

(46) *RevierGlockenChor Bottrop* (Bottrop, DE-NW)

(47) *Rezound!* (Blue Springs, US-MO)

(48) *Queen City Bronze* (Highland Heights, US-KY)

(49) *Raleigh Ringers* (Raleigh, US-NC)

(50) *Ring of Fire* (Hillsboro, US-OR)

(51) *Michèle Diana Sharik* (Piñon Hills, US-CA)

(52) *Sonos Handbell Ensemble* (Berkeley, US-CA)

(53) *Carla and Larry J. Sue* (Holland, US-MI)

(54) *Southminster Ringers* (Pittsburgh, US-PA)

(55) *Three Rivers Ringers* (Pittsburgh, US-PA)

(56) *Timbré Ensemble* (Southern California, US-CA)

(57) *Voices In Bronze* (Frazier Park, US-CA)

(58) *Westminster Concert Bell Choir* (Lawrenceville, US-NJ)

And these groups do not wear gloves:

(1) *Allegro Handbell Ensemble* (Hinsdale, US-IL)

(2) *Brookhampton Bellringers* (Brookhampton, AU-WA)[67]

(3) *Cathedral Bells of St. John's Lutheran Church* (Orange, US-CA)

(4) *Circle City Ringers* (Otterbein, US-IN)

(5) *Dorothy Shaw Bell Choir* (Fort Worth, US-TX)[68]

(6) *Rooke Chapel Ringers* (Lewisburg, US-PA)

(7) *Strikepoint* (Duluth, US-MN)

(8) *Philadelphia Handbell Ensemble* (Hatfield, US-PA)

(9) *Twin Cities Bronze* (Lakeville, US-MN)

(10) *Virginia Bronze* (Alexandria, US-VA)

(11) *Vivace* (Aibonito, PR)

These lists may not be representative for all handbell ensembles in general, but they may show that there are acclaimed ones in both categories.

4.4.3. On eyeglasses

Musicians have to be able to read sheet music so they have to use corrective lenses if necessary. The collection [Sch04, *Prescription glasses*, *retrieved* June 8, 2013] gives some ideas on special *handbell glasses*.

4.5. Arranging on stage

Most ensembles do not have access to a *road crew* or to *roadies*, hence they have to build up tables, music stands and so on themselves. Ringers may examine local features and ponder about the best setup or simply rethink their usual setup to improve their setup from time to time.

In some ensembles every ringer has her or his own copy of sheets and, therefore, everyone has her or his own music stand. The right place for

[67] Known for being founded in 1904 and using the same set of bells since then, compare [Bro04, *Centenary Celebrations*, *retrieved* Februar 24, 2017] and the video [Wil13].

[68] Known for not using any tables, sheet music nor a conductor on performances, compare [Dor14, *Index*, *retrieved* Februar 24, 2017].

music stands is not discussed here. Two aspects naturally determine a music stands position: First every attributive ringer should be able to read the sheets without any difficulty, and second the position (and height) should not affect the communication between the conductor and the ringer. [Ber12, pages 150-151] discusses music stands.

We focus on the arranging of a handbell ensemble only and give some basic advice.

Besides putting tables in one straight line (and putting several lines in one row), there are the following basic types: *V-tables* which can be seen as a generalization of straight lines, *U-tables*, *G-tables* which is a modification of U-tables and *S-tables* which is a further modification of G-tables. In figure 19 one can see these commonly used basic types.[69] This classification is not limited to table size in general, and hence is not limited to the number of handbells which can be used. The size of the bells sets is just a hint. Also one might alter the angles. In [Joh87, page 13] one can see U-tables with obtuse angles instead of right angles. The choice of configuration is affected by the number of instruments used and preferences of the ensemble and is limited by the size of the stage. It is clear that an ensemble should arrange tables in rehearsal similar to the concert, so the leader can make sure the ensemble can work with the setup.

Putting some tables in a row has one major disadvantage. It becomes difficult for the director to communicate with individual ringers since some gestures or rather their direction lose their definiteness.

Handchimes and handbells can be placed on the same tables or on different ones. For example, handchimes can be placed on V-tables in the center of G-tables where the handbells are set on, see figure 20. This implies that the ensemble consists of one handbell choir and one handchime choir. The convexity of G-tables or S-tables is for large bass handbells.

American standard tables for handbell ringing measure about $3\,\mathsf{ft} \times 2.5\,\mathsf{ft}$ and are adjustable in height. One note to the height of the tables: the table, or rather handbells lying on the tables, should not limit ringers. The table's height should make picking up instruments easy, especially for bass

[69] Note that the conductor turns his back to the audience. The audience is able to see the ringers. In general the arrangement cannot be chosen freely since an ensemble has to deal with local features.

ringers. Ringing a handbell should not be affected by a possibility of striking instruments set up on the table.

4.6. Handbell setup

During performances a handbell usually is at one of these places: in a ringer's hand, on the table or put aside, for example on a pad lying on the floor, if the handbell is not needed in a specific piece. From a conductor's view, the handbells are set on the table from low pitch (right) to high pitch (left).

As usual the keyboard setup inspired by keyboard instruments, see figure 21a, can be seen as a starting point for setting up. Some pieces are a lot easier to ring when the setup is adapted for individual pieces or individual parts of pieces, therefore one ringer does not have to place his or her handbells in chromatic order. An arc setup, see figure 21d, for large bells has proven to be practical, see also [Sue07, page 39].

Higher pitched handbells can be set up like two manuals atop of each other, see figure 21b, or doubled, see figure 21c, which may come in handy for multiple handbells in hand techniques like shelley ringing. When handchimes are used along with handbells, the handchimes are usually set up similar to figure 21b or figure 21c depending on personal preferences and handchime dimensions.

The layout of non-diatonic handbells may be restricted to non-bass-handbells due to their dimensions.

Larry J. Sue introduces a method for a layout using simple graph theory in [Sue07, 15.3 *Layout*]. This technique is not (necessarily) restricted to bass ringers and is designed for ringers who have to ring a larger number of bells, thus making this method interesting for solo ringers, too. He also includes two examples with twelve and eight bells, respectively.

4.6.1. A setup example: Bach's passacaglia in Cm

To introduce the problem of a proper bell setup, a real life example is given: When M. Jedamzik rings[70] Johann Sebastian Bach's passacaglia in C minor, BWV 582, (see figure 17), a chromatic setup would be C_2, F_2, G_2, B_2. Since all handbells have to be rung in a row in pattern B_2, F_2, G_2, C_2 (see figure 22), a *weaving* technique is necessary. Another possible setup is B_2, G_2, F_2, C_2 representing ringing order *1 3 2 4*, so a left hand ring is followed by a right hand ring and vice versa, see figure 21d. The handbells F_2 and G_2 may be damped on the table.

4.6.2. General patterns

A true benefit of the pattern as in section 4.6.1 is to avoid crossing arms. There are similar patterns when more bells are involved. The pattern layout probably debuted in solo ringing but is considered as a valuable tool for advanced ringers. Three of those patterns are:

separated hands First all uneven and then all odd numbers: this is a simple sequence of separated even (right hand) and odd (left hand) numbers, both sorted ascending. This reduces the ways of both hands, see table 13a. The pattern can be extended and reduced (by leaving out the last numbers, which would give $(1, 3, 2)$ or $(1, 3, 2, 4)$ for example) naturally. This pattern is limited by the fact that the ringer may have to move his entire body in a zigzag pattern making this pattern unusable for large numbers of bells, especially with large bells which need even more horizontal space.

travel right Another classic pattern is more elaborate, but not too difficult. The even numbers are sorted ascending, where the even number k is placed between the odd numbers $(k + 1)$ and $(k + 3)$ and 1 is placed in first position, see table 13c. This pattern can be extended and reduced, too. By switching from one bell to the next, each hand has to "jump" over one bell and grasp the next one. The clear benefit is

[70] He has not really rung this nice but fuga-missing transcription by Kevin McChesney yet, since the Handglockenchor Wiedensahl has neither a lower seventh octave nor the sheets.

that the ringer moves from left to right instead of the zigzag. This sequence is actually known as [Slo64, sequence A065190].

travel left Just as travel right, but with inverse order, see table 13e.

Both patterns are given with left hand or right hand starting. The odd numbers are to ring with starting hand and the even numbers are to ring with the other. Note that the separated hands pattern and the travel right pattern are equal for a maximum of 4 bells. Note that these patterns are optimized for one particular sequence of bell changes with n bells without repetitions (as mathematical objects: elements of the symmetric group $\mathfrak{S}_n$ and not $\mathbb{Z}_n^m$). The listed patterns might not be an adequate setup for a whole composition.

4.6.3. Handbell cradle and handbell damp-bar

Bass handbells may be placed on extra shapes made of foam, see figure 23a to stabilize them on the table. These objects called *handbell cradles*, see [RR09, *Merchandise. Bell Cradles*, *retrieved* June 28, 2013] also make table damping with bass handbells more effective.

A similar concept is a *handbell damp-bar* made of softer foam, see figure 23b. Several handbells can be placed on a handbell damp-bar in a row depending on the dimensions of handbells and damp-bar. The foam needs to be softer to allow handbell to deform the foam due to its weight. Due to its shape, a damp-bar can be covered easier with fabrics to match the used table cover than a handbell cradle.

Nancy Kirkner defines a *handchime pad* in [Kir11, *Handbell Articles (Blog). Chimes in solo work from August 29, 2014*, *retrieved* September 19, 2013] as a rectangular cuboid made of foam. When the handchime tine opposite to the clapper rests on the handchime pad, a space below the handchime tube is given, which allows picking up handchimes easier. When the tine next to the clapper rests on the chime pad, the clapper rests on the table and the chime can be played (gently) with a mallet. The article also features other interesting thoughts on handchimes not listed here.

4.7. Web 2.0

The widespread use of the world wide web and the huge amount of user-generated contents have an influence on the world of handbells, too. A director may find video clips of other ensembles performing a new work of his favor and so the director orders sheets afterwards. Sound samples or sheet excerpts on publishers' web pages may also help a director.

The following translations might help finding video streams on the net: The German word "Handglocke" literally means *handbell* and is used by German handbell ringers. The term "Tonstab", which translates to "sound rod", is used along "handchimes" and manufacture depending terms. The Japanese loanword "ハンドベル" translates to *handbell*. Likely the Chinese word "手鈴" translates to *handbell*.

Some directors do arrange pieces for handbells and may also publish them on the internet.

Ringers may prepare for rehearsals by reading sheets while listening to corresponding sound files or videos respectively on the internet via social media web pages, composers' web pages or publishers' web pages, see section 5.1.1. Another option to train difficult rhythms is to return them in a scorewriter software such as *MuseScore*, see [MB14],[71] and listen or even better phantom ring with the audio output.

There still are more classic ways of bidirectional communicating by text via message boards, discussion groups or the Usenet.

It should be noted here, as well, that all users should always remember netiquette.

4.8. Ringers' Ten Commandments

A classic among ringers are the ten following rules which are roughly inspired by Christians' Decalogue as in the King James Bible. The original author of the ringers' commandments as presented in figure 24 is unknown.

[71] There are also a few compositions for handbells on the server; Just search for "handbells" on the website. The community also provides a tool for counting bells which is useful when making the handbell used chart.

4.9. Etiquette

A more serious attempt compared to section 4.8 for etiquette is made by [Mon03, *Rehearsal/Performance Etiquette*, *retrieved* April 3, 2013], [Cam97, *retrieved* June 5, 2004] and [EY09, *A Rehearsal Etiquette Guide. August 3, 2009*, *retrieved* April 3, 2013]. All three texts[72] are merged and adjusted for a handbell ensemble, excluding the part for the director:

4.9.1. General thoughts

(a) Having a good attitude can get you through a lot of rough times.

(b) Remember that while striving to be "perfect" our true goal should be to make great music and behave like humans should.

(c) If each ringer and conductor commits to the process of rehearsal together, they can turn sheet music into real music!

(d) Your reputation develops and follows you very closely. Your work and how you approach your work and how you treat your colleagues all matter. It is all carefully watched. Nothing you do in rehearsal is in isolation but is always in relationship to others and should be treated that way.

4.9.2. Continual etiquette

(a) Always keep your head.

(b) Follow the corresponding care and maintenance instructions.

(c) Arrive on time. Understand your schedule: Some conductors understand the beginning of a rehearsal as the first tone, that is the ringers have to set up and be ready at the beginning of the rehearsal.

(d) Come prepared. Study your music.

[72] The additional text [Kea00, *Rehearsal Hints*, *retrieved* June 6, 2004] also covers aspects for a director and the additional text while the text [Ber12, pages 155-157] focuses on the preparation related to healthy ringing.

(e) Behave.

(f) Be respectful even if you think that others are not.

(g) Aside from music, be quiet.

(h) Pay attention. Listen to your conductor's instructions so that he does not have to tell you twice.

(i) Do not stare at one of your colleagues when you are not playing.

(j) In the same vein, never look over at someone after he or she has made a mistake! Do not grimace, laugh, shake your head, or anything else either. In other words, do not react!

(k) Some pieces require different instruments which may have to be tuned. When the tuning note is given, stop making noise.

(l) Do not interrupt or hinder by using a mobile phone.

(m) Be prepared for the next piece. This includes getting your gloves on early enough.

(n) Dress yourself appropriately, compare section 4.4.1. Your dress code also reflects your respect of the event.

(o) Handle the instruments with care.

(p) No chewing gum.[73]

(q) Get familiar with the pieces you ring. You might want to take the sheet music home to get to know your pieces.

[73] From [OnL01, *Chewing Gum*, *retrieved* August 10, 2001]: "The primary reason for not doing so in not just the way it might look to observers, it is the fact that no matter how careful you are, very small particles of spittle [original: 'spital' — M. Jedamzik] will make its way to the bells being rung by the person chewing gum. The fact is the juices from you [sic! — M. Jedamzik] gum and mouth are much more corrosive than the oil from your hands. Very small spots, left uncleaned [original 'un cleaned' — M. Jedamzik] will eat away the finish on the highly polished handbells and destroy their beauty. In extreme cases it can eat into the metal and hurt the sound of the bell."

(r) Despite hygienic objections, a handshake is still a more common ritual. Although a handshake is performed without gloves, it has become common sense that a handshake with gloves is not offensive among handbell people.

4.9.3. Pre-rehearsal and pre-performance etiquette

(a) Help all members to set up. This includes transportation of instrument cases, setting up tables and handbells as well as note stands. This also includes putting your personal property like jackets and handbag in the right place.

(b) Check your instruments for damage or need for readjustment. Fix them by yourself if you are able to or report them.

(c) Move the clapper of handbells and especially chimes once so that the clappers operate smoothly.

(d) Do an "idiot check" before you go to rehearsal. Do you have your sheet music? Gloves? Assignment Notes? Handbell Compendium?

4.9.4. Rehearsal etiquette

(a) Do not ask stupid questions. Do not get into a debate with a conductor or challenge a conductor during rehearsal. Make notes and ask questions on your own time.

(b) The conductor is the boss. If he or she asks you to play something up an octave, do it. If you are told to change a note, do it.

(c) If there are sections you experience problems with, you can kindly ask the conductor to repeat the section once more to show the problem and resolve it.

4.9.5. Performance etiquette

(a) Do the "idiot check". Twice.

(b) Arrive in plenty of time.

(c) Do not wear anything that causes you to stand out more than anyone else on stage.

(d) If you notice that a handbell might be damaged and you are unable to fix the problem and ringing the handbell again might lead to additional damage, do not ring the handbell again, especially during a concert.

(e) Do not talk during performances. “Talk” includes counting measures of rest out loud.

(f) Do not tap your foot. A toe is fine. A foot is noisy and can also be distracting.

(g) Do not grimace or shake your head or react in any other way when you make a mistake!

(h) Do not change your sheet music or prepare for the next piece until the audience has finished applauding.

(i) Your conductor may bow after every performed piece. All ringers should bow altogether when your conductor gives a sign to do so.

(j) If you notice that you are hurt, you should consider stopping your ringing. Especially when you are on a concert tour, missing one concert is better than missing the remaining tour.[74]

(k) If you missed ringing a handbell at the right time, do not ring it at the wrong time.

(l) If your mallet, handbell, notebook or whatever falls to the floor, pick it up right away. If you do not, your audience will not see or hear anything until you do. Do not prolong their misery.

4.9.6. Post-performance etiquette

(a) Answer listeners’ questions and be polite. Do not make the performance look bad by talking about mistakes. Do not make your general music look bad by being overwhelmed (too much).

[74] It is difficult to give right and general advice.

(b) If listeners are allowed to try ringing handbells, instruct them and tell them not to touch the bronze and make sure they do not drop or mishandle any instruments.

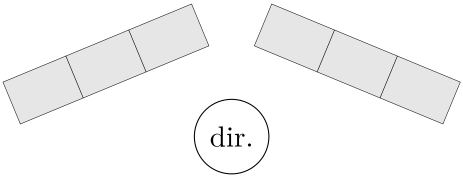
(c) Help all members to dismantle the setup. This includes transportation of instrument cases, tables and handbells as well as note stands. Check instruments for damage again. Follow the corresponding care and maintenance instructions.

(d) Make sure that nobody forgets equipment or personal goods when leaving the performance space.

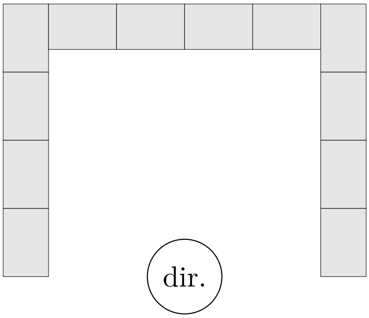
4.9.7. Post-rehearsal etiquette

(a) Help all members to dismantle the setup. This includes transportation of instrument cases, tables and handbells as well as note stands. Check instruments for damage again. Follow the corresponding care and maintenance instructions.

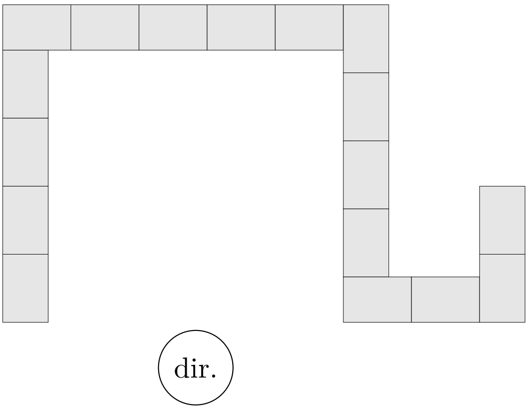
(b) Take your sheet music with you in case you want to study at home.



(a) *V-tables*: stage plan for 3 octaves

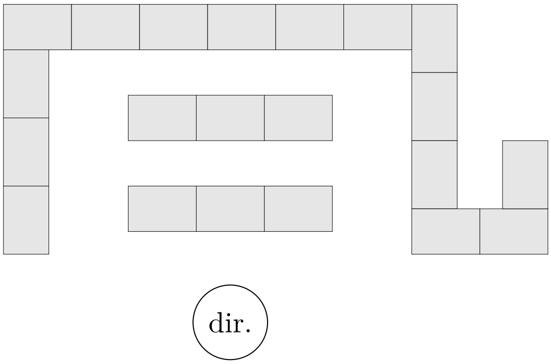


(b) *U-tables*: stage plan for 5 octaves



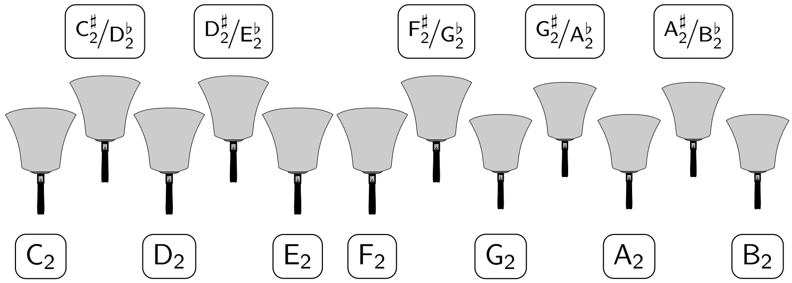
(c) *S-tables*: stage plan for 6 octaves. For *G-tables*, most right tables are missing

Figure 19: Stage plans of the three basis types

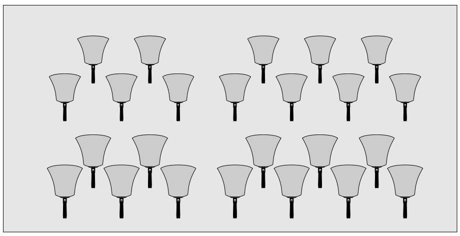


This modification is for 6 octaves of handbells and 5 octaves of handchimes. S-tables for handbells and two straight V-tables for handchimes. The *Handglockenchor Wiedensahl* used a similar arrangement on a concert in 2016.

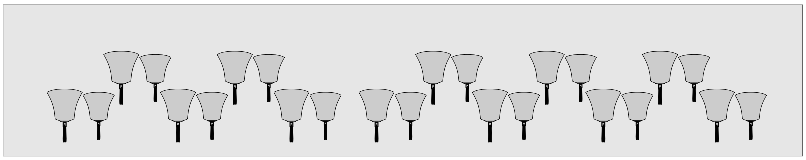
Figure 20: Stage plans modification example



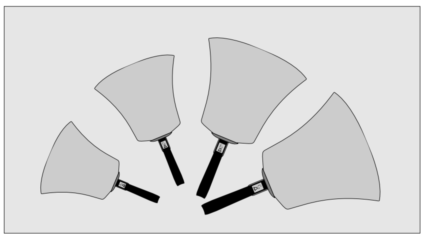
(a) keyboard setup (here: lower seventh and lower sixth octave)



(b) manual setup for treble handbells

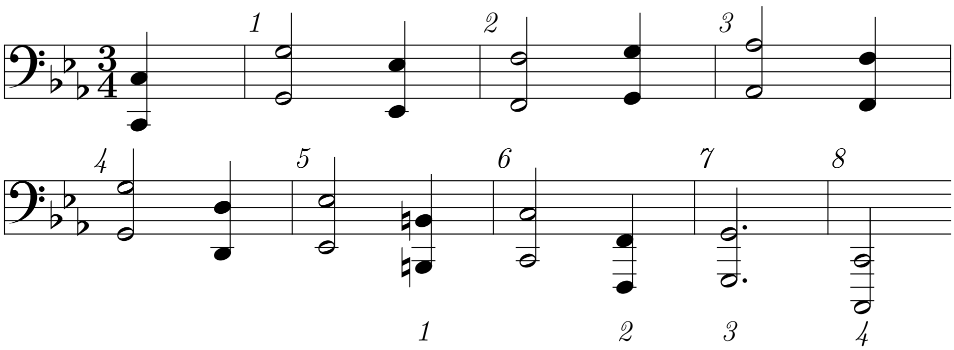


(c) doubled manual setup for treble handbells



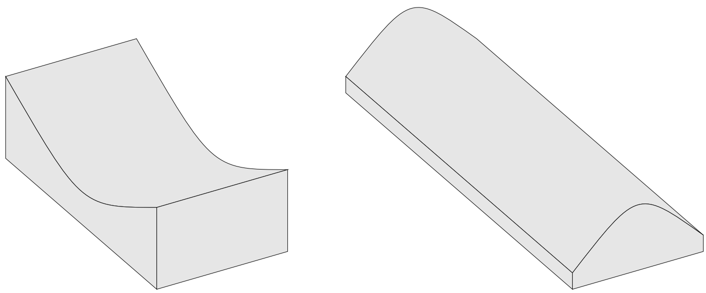
(d) arc setup (especially for bass bells, here from left to right: B_2, G_2, F_2, C_2)

Figure 21: Examples for bell setup



In contrast to the original, the bass line here is doubled.

Figure 22: Excerpt from J. S. BACH's passacaglia in C minor, BWV 582



(a) A handbell cradle for a single bass handbell

(b) A handbell damp-bar

The figures are not scaled to the same factor.

Figure 23: Bass handbell damp-pads

left hand	1	3	5	7	9				
right hand						2	4	6	8
setup	1	3	5	7	9	2	4	6	8

(a) setup: separated hands (left hand starting)

left hand	2	4	6	8					
right hand					1	3	5	7	9
setup	2	4	6	8	1	3	5	7	9

(b) setup: separated hands (right hand starting)

left hand	1	3		5		7		9	
right hand			2		4		6		8
setup	1	3	2	5	4	7	6	9	8

(c) setup: travel right (left hand starting)

left hand	2		4		6		8		
right hand		1		3		5		7	9
setup	2	1	4	3	6	5	8	7	9

(d) setup: travel right (right hand starting)

left hand	9	7		5		3		1	
right hand			8		6		4		2
setup	9	7	8	5	6	3	4	1	2

(e) setup: travel left (left hand starting)

left hand	8		6		4		2		
right hand		9		7		5		3	1
setup	8	9	6	7	4	5	2	3	1

(f) setup: travel left (right hand starting)

The first two lines of the table just show the structure of the patterns and do not indicate a necessarily vertical offset, although this might make switching the instruments easier.

Table 13: Patterns for handbell setup

I Þou ſhalt attend þy rehearſals wiþ ſteady faiþfulneſſe.

II Þou ſhalt not touch gravene metal wiþ þy bare hands.

III Þou ſhalt not take þy ſharps and flats in vain.

IV Remember þy performance dates and keep þem holy.

V Honor þy director þat þy days may be long vpon þe lande.

VI Þou ſhalt not covet þy neighbore's bell aſſignments.

VII Þou ſhalt not tap þy foot wiþ exceeding loudneſſe.

VIII Þou ſhalt not ſteal þy neighbore's pencil.

IX Þou ſhalt return þy bells to þeir caſe, ſhiny and vnſcaþed.

X Þou ſhalt return þy director to his car, ſhiny and vnſcaþed.

Note the use of the letters *long-s* ſ and *thorn* þ. The original King James Bible [Bib11] restricts the use of thorn to abbreviations.

Figure 24: Ringers' Ten Commandments

Labor volvptasqve, dissimillima natvra, societate qvadam inter se natvrali svnt ivncta.

from Ab Vrbe Condita Libri (liber V)
Titvs Livivs

5. Notation

Before giving an overview on the techniques, some ideas on sheet music are presented.

5.1. On sheet music

Sheet music is an important and valuable part of a handbell ensemble. It is to be handled with care.

5.1.1. Dealing with sheets

New sheets can be considered as raw. They have to be prepared with notes and annotations. Inscriptions are not for the forgetful ones only, but also for posterior concerts.

A handbell ensemble has several copies of sheet music for a piece. It is reliable to classify each of these copies by the ringers using this specific copy. For example: If every ringer has his or her own copy and the ensemble uses the assignment strategy given in section 3.5.2, the copies can be enumerated by the corresponding position number. An ensemble can still use this pattern even if it does not use this assignment strategy and has not that many copies by simply omitting the odd numbers, resulting in a practice-proven number of copies.

Some works are upgraded by the musical director of a handbell ensemble with some additional octaves of bells and some instructions, for example on articulation. In general it is a good idea to write down these changes in the sheets. When some ringers note their own annotations a pencil is recommended, so notes can be erased and the sheets can be used by other replacing ringers. Marks with different colors may be be limited by color blindness which requires attention when introducing ringers to new bells. [Ber12, page 151] sees *Color-coding* as a valuable tool for beginners and ringers, who have lost some of their visual capabilities.

Some ensembles work with one sheet per ringer so every ringer can make his or her personal notes.

Preparation by reading and preparing sheets and listening to records should be considered by every ringer, especially after the first rehearsals (when problems show up) or before the first rehearsal. The latter consideration may help the initial ringing experience.

As mentioned before a pencil is recommend for writing on sheet music. Another idea based on [AHST95, page 20f] is to disassemble the sheet music, to put each page in a sheet protector and to put this in a file folder. Be aware of possible reflections. By disassembling one has to make sure that bibliographic data of the piece is noted on each page. The ringers can then use water soluble marking pens in different colors on the sheet protectors. According to [McC12] which is part of the series [McC16] this procedure might be copyright infringement. Turning snippets as in section 5.1.2 can still be used.

In addition to the traditional way of obtaining sheet music by purchasing physical copies, the internet made another channel of distribution possible, where ensembles buy licenses to print sheet music of themselves and replace worn out copies without additional license fees. One of those distributors which focuses solely on this model is [STEP16] which was founded by Kevin McChesney and Michael Kastner and is now owned by Megan Reishus.

5.1.2. Turning snippets

Turning a page of sheet music while ringing might by difficult, especially when wearing gloves. A set of *turning snippets* attached to each sheet might make turning pages easier and may also help to avoid skipping pages.

The turning snippets are simply Scotch-taped on the outer side of the sheet music, see figure 25. The snippets themselves can be blank paper or notes on the next measure of the following page. If several ringers use the same copy of sheet music, the ringers can write the name of that person on the turning snippets who can turn that particular page best.

Some ensembles also use clothespins instead of turning snippets.

5.2. Common notations

A huge set of musical terms was invented in order to indicate the actions intended by the composer, especially when it comes to tempo. In this section we recall the handbell-specific notations and a small selection of notations used in western music. Since the use of anything that lies beneath the notes can be considered as interpretation, the conductor has to give instructions to the ringers.

There are notations which are sometimes listed as techniques. The terminology in section 5.3.2 can be seen as notations. Such terminology helps in understanding the score and noting setup makes ringing simpler.

5.2.1. Voice-leading lines

"The movement of any voice, melody, or other line from one staff to another may be clarified by the use of voice-leading lines, [see figure 26 for an example — M. Jedamzik]. When possible, the voice-leading lines should extend from note head to note head. For additional clarity, rests may be added," [HMA10]. This notation is likely used within C_4 to C_6.

5.2.2. An example for damping

The more or less unusual example with respect to beat 3 of bar 2 in figure 27 is inspired by [OnL01, *Handbells a Basic Tutorial. #4 Ringing Musically*, *retrieved* August 2, 2001]: The ringer who rings C_3 on figure 27 has to ring it on beat 1 and 4 of measure 1, on beat 1 and 3 of measure 2. The ringer has to damp C_3 on beat 3 of measure 1 and beat 1 of measure 3 only. The lucky ringer who rings C_2 has to ring it on beat 1 of measure 2 and has to damp this most beautiful handbell at beat 4 of measure 3.

This aspect of notation may naturally appear when two musical lines cross.

A relatively uncommon view on damping is proclaimed in [Woo89]. Dale Wood advocates the sole use of the *let vibrate* and the departure from damping which also leads to a simpler notation. This simpler notation makes understanding the music more difficult since information is lost.

5.2.3. Dynamics

The common ways of defining derivatives of the dynamics as in tables 14 to 17 are:

(a) The change of dynamics is often given by a decrescendo or crescendo which is specified with a dynamic symbol at the end of the change of volume.

(b) A small "s" in front of the dynamic notations means "subito", and means that the dynamic is to be changed to the new notation rapidly.

(c) For two different dynamics x and y the notation xy means a rapid change to x and immediately to y. Commonly used is *fp*, named *forte-piano* .[75]

5.2.4. Repetitions

This section is based on [SW01, *Coda (music)*, *retrieved* February 1, 2014], [SW01, *Dal Segno*, *retrieved* February 1, 2014], and [SW01, *Da Capo*, *retrieved* February 1, 2014]. There are various symbols to define repetitions in sheet music. A simple example for each notation is given to illustrate.

[75] There are limits of this notation: For example the notation of *fff* for a rapid change from *ff* to *f* is a bad idea since this notation is unambiguous, the notation *ff-f* might be easier to understand.

Repetition Sign and *Volta bracket.* The repetition sign (a bold bar with two dots) indicates which bars are to be repeated. This is often combined with the Volta bracket above the staff: The numbers in the brackets indicate which segment is to be played in which repetition. Compare figure 28.

Da Capo. Means “from the beginning”: Repeat every bar from the beginning. Compare figure 29a.

Da Capo al Fine. Means “from the beginning to end”: Play the whole piece as written and then begin at the beginning and stop at *Fine.* Compare figure 29b.

Da Capo al coda. Play from the beginning to *D.C. al Coda*, play from beginning to indicated bar and then jump to the coda. Compare figure 29c.

Dal Segno al Fine. Play from the beginning to *Dal Segno al Fine*, repeat back to the sign, and end the piece at the measure marked *Fine.* Compare figure 30a.

Dal Segno al Coda. Play from the beginning to *Dal Segno al Coda* and then repeat back to the sign, and when *al Coda* or *to Coda* is reached jump to the coda symbol. Compare figure 30c.

5.2.5. Tempo

Various tempo markings are an important aspect of musical notation. In the handbell world they usually can be simplified as “Keep an eye on your conductor!”

5.3. Handbell techniques and their notation

This section is based on [Jef10, *Handbell Notation Symbols & Definitions*, *retrieved* April 2, 2012] and therefore on [HMA10]. Remember that sheets may use different fonts, so the symbols in sheet music may slightly differ from the symbols listed in this section.

The techniques in this section are described briefly: The description only implies how to generate the tone, not how to generate the tone in a proper or efficient way. For tips with bass bells and the healthy-ringing aspect, the books [Sue07] and [Ber12] are recommended as well as a handbell clinician's advice.

5.3.1. Basic terminology

The tables 18 to 30 lists the most common ways of playing with handbells and their notations. Some techniques like *water bell*, which roughly means to tip the casting of a sounding handbell partially into water[76], or *bowed bell* or *bowed handchime*, which roughly means to draw a violin bow on a (suspended) handbell casting or handchime tine, are not listed and may be explained or specified by footnotes in sheets.

Sometimes notations may be altered in sheet music for simplicity or clearness. For example a dot below or a on top of a note head often is used for staccato techniques like *pluck* or *mallet on table*.

Some of the techniques may not be used by some choirs due to lack of material.

5.3.2. Additional information and ensemble terminology

There also are some notations for switching bells and layout, see table 31. Notations for multiple bells in hand ringing are given in table 32. Both notations are taken from [Jef10, *Solo and Ensemble Notation Symbols*, *retrieved* April 2, 2012] and is therefore based on [HMA10].

The standard notation for the impressive eight-in-hand technique as used by Danny Lyons is unknown yet. There is also a technique for direct in hand flipping a four-in-hand to shelley and back known as *Fred flip* by Fred Snyder, see [Sny09]. [Ber12, pages 93-95] describes the *combo-ring* or *four-in-hand lift* which also enables a skilled ringer to use the four-in-hand setup to ring both bells simultaneously without changing the setup.

When using multiple bells in hand technique, the bells usually are named *primary bell*, *secondary bell* and so on, see [Ber12, page 88], where the

[76] Less aggressive and still usable fluids presumably exist.

primary bell is the one closest to the palm of the hand and usually the heaviest. This nomenclature extends in natural order.

[Ber12, page 95] does not recommend the six-in-hand technique. According to [Sha14], this is based on an old technique of six-in-hand and a proper six-in-hand technique is compatible with healthy ringing.

Figure 25: Turning snippets attached to sheet music

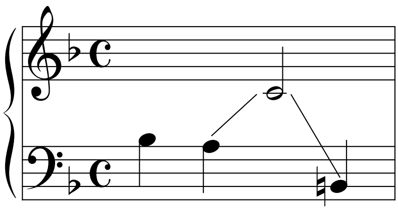


Figure 26: An example for voice-leading lines

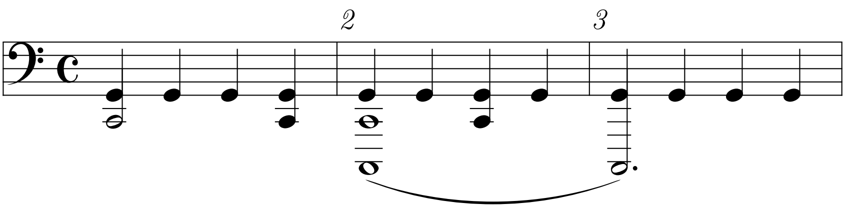


Figure 27: An example for damping

pictograph	name	description
ppp	pianississimo	Extremely soft. Very infrequently does one see softer dynamics than this, which are specified with additional *p*s and resulting in additional “iss”-syllables in the name or *n-times piano*. The level *ppp* is sometimes named *piano pianissimo*.
pp	pianissimo	Very soft. Usually the softest indication in a piece of music.
p	piano	Soft.
mp	mezzo piano	Literally, half as soft as piano.
mf	mezzo forte	Literally, half as loud as forte. If no dynamic appears, mezzo-forte is assumed to be the prevailing dynamic level.
f	forte	Loud.
ff	fortissimo	Very loud. Usually the loudest indication in a piece.
fff	fortississimo	Extremely loud. Very infrequently does one see louder dynamics than this, which are specified with additional *f*s and resulting in additional “iss”-syllables in the name or *n-times forte*. The level *fff* is sometimes named *forte fortissimo*.

Table 14: Notations for sound volume

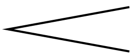
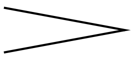
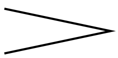
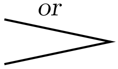
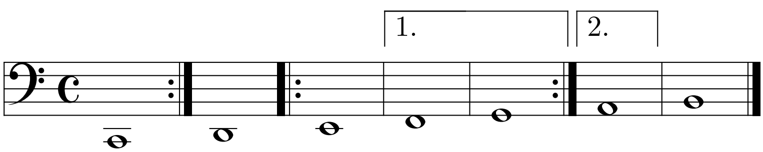
pictograph	name	description
	crescendo	Gradual increase in volume. Can be extended under many notes to indicate that the volume steadily increases during the passage. The amount of increment can be specified with the usual volume indicators.
	decrescendo	Gradual decrease in volume. Can be extended under many notes to indicate that the volume steadily decreases during the passage. The amount of increment can be specified with the usual volume indicators. Also named *diminuendo*. Not to be confused with *accent*. See also the handbell technique CD (controlled diminuendo).
 n *or*  ∅	niente	The word *niente* literally means “nothing” and means that the loudness is decreasing until the tones are inaudible. This can be used on handbells at the end of a part by not damping the sounding bells for a while.

Table 15: Notations for continuous volume change



The order to ring in this example is C_3, C_3, D_3, E_3, F_3, G_3, E_3, A_3, B_3.

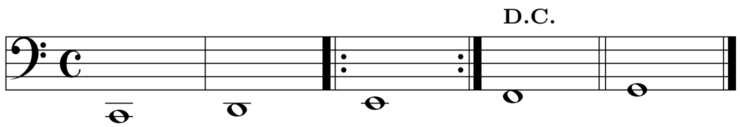
Figure 28: Examples for repetitions: repetition sign and Volta bracket

pictograph	name	description
sfz	sforzando	Literally “forced”, denotes an abrupt, fierce accent on a single sound or chord. When written out in full, it applies to the sequence of sounds or chords under or over which it is placed.
>	accent	The note is played louder or with a harder attack than surrounding unaccented notes. Not to be confused with *decrescendo*.
∧	marcato	The note is played somewhat louder or more forcefully than a note with a regular accent mark.

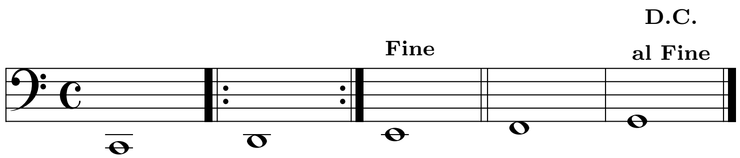
Table 16: Notations for accents

pictograph	name	description
𝄐	fermata	An indefinitely-sustained note, chord, or rest. Usually appears over all parts at the same metrical location in a piece, to show a halt in tempo. It can be placed above or below the note. In the latter case the symbol is rotated by $180° = \pi$. The piece is usually continued on the sight of the conductor.
//	caesura	Indicates a brief, silent pause, during which time is not counted. The piece is usually continued on the sight of the conductor.

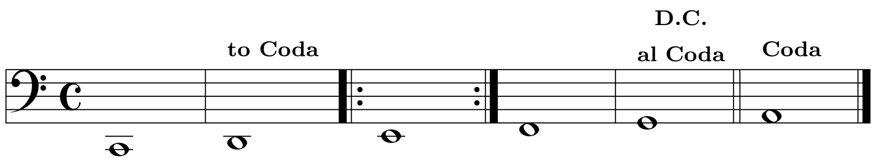
Table 17: Notations for fermata and caesura



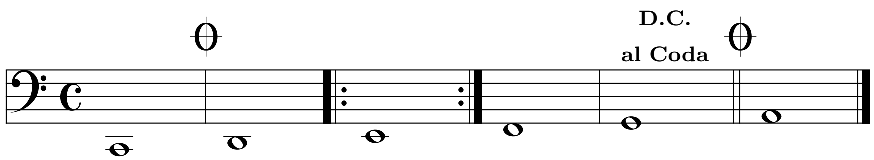
(a) example for Da Capo: C_3, D_3, E_3, E_3, F_3, C_3, D_3, E_3, E_3, F_3, G_3



(b) example for Da Capo al Fine: C_3, D_3, D_3, E_3, F_3, G_3, C_3, D_3, D_3, E_3

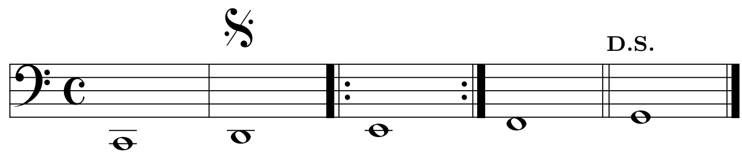


(c) example for Da Capo al Coda: C_3, D_3, E_3, E_3, F_3, G_3, C_3, D_3, A_3

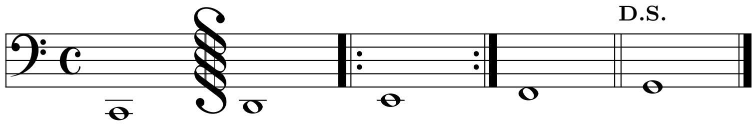


(d) another notation for Da Capo al Coda: same as figure 29c

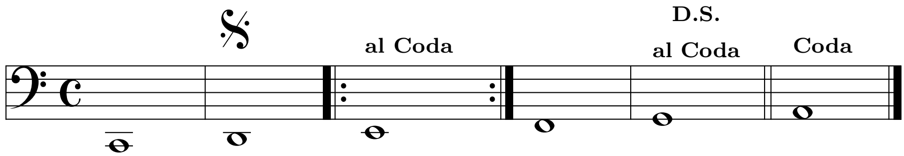
Figure 29: Examples for repetitions: Da Capo



(a) example for Dal Segno al Fine: C$_3$, D$_3$, E$_3$, E$_3$, F$_3$, D$_3$, E$_3$, E$_3$, F$_3$, G$_3$



(b) another notation for the Dal Segno symbol: same as figure 30a



(c) example for Dal Segno al Coda: C$_3$, D$_3$, E$_3$, E$_3$, F$_3$, G$_3$, D$_3$, E$_3$, E$_3$, A$_3$

Figure 30: Examples for repetitions: Dal Segno

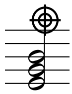
pictograph	**name**	**description**
	damp	“The *damp sign* indicates the cessation of sound in let vibrate passages,” [HMA10]. The terms *étouffé*, *target*, or *bullseye* are also used.
	selective damp	“The *selective damp* symbol indicates that only the handbells represented by the cue-size notes should be damped. A damp sign incorporated in the stem of a chord indicates the selective damping of that chord. A damp sign incorporated in the stem of a single full-size note indicates the selective damping of that note,” [HMA10].
TLD	table land damp	“To achieve a *table land damp*, damp the handbell by pressing it mouth-down into a padded table on a given beat. This technique may not be possible on handbells with clappers projecting beyond the lip of the handbell. An explanatory footnote should be used,” [HMA10].

Table 18: Notations for damping

pictograph	name	description
LV	let vibrate	"The term *let vibrate* means *Let Vibrate* or *Laissez Vibrer* (in French), allowing handbells to resonate regardless of note values or rests until damping is indicated. A let vibrate symbol if placed above the treble or below the bass staff, applies to that staff only. A let vibrate symbol centered between staves applies to both staves," [HMA10]. *Let vibrate* can also be terminated with *damp*, *ring* or another technique. If a LV-symbols follows another, this usually means that the previous bells rung in let vibrate are to damp if they are not to ring again.

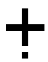
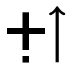
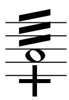
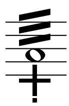
Table 19: Notations for let vibrate

pictograph	name	description
SB	singing bell	"The *singing bell* is accomplished by using an eight inch wooden dowel of at least 1 inch in diameter or more, and covered either with suede or leather or dipped in a plastic coating, in the same manner *heat shrinking tubes* can be applied more easily. The rubberized shaft/handle of some of the larger bell mallets also works well. Lightly tap the lip of the bell to start the sound, then, quickly rub the rim of the bell with the stick in a clockwise motion to begin the 'singing'. Some degree of pressure is required. For more information, visit the internet and search for 'singing bell'. Soft suspended mallet rolls may be used in place of singing bell, if necessary," [HMA10]. This technique is based on the Asian instrument *singing bowl*.

Table 20: Notations for singing bell

pictograph	**name**	**description**
	echo	"To perform the *echo* technique, the handbells are rung on beat one and then lightly but precisely touched to the padded table on the counts indicated," [HMA10].
$\circlearrowleft$	gyro	"To execute a *gyro*, ring the handbell, while holding in the vertical position slowly rotate the handbell to produce a slight vibration in sound," [HMA10].
vib.	vibrato	"To achieve the *vibrato* effect, ring the handbell, gently move the handbell from side to side using the wrist, not the arm, to produce a wavering sound," [HMA10].
Sw *or* $\uparrow\ \downarrow$	swing	"A *swing* indicates a full-arm swing after ringing the handbell. A *Swing* symbol and/or arrows are used to indicate swings. Arrows should be synchronized with the beats on which the swings occur. Numbers may be used to specify the beats on which the swings are made," [HMA10].

Table 21: Notations for vibrato effects

pictograph	name	description
Mallets	mallets	"The *mallet* technique can be used either on a suspended handbell held by the handle and struck with a mallet or a handbell that is resting on a padded table and struck with a mallet," [HMA10].
	suspended mallet	"This symbol indicates that a suspended handbell is held by the handle and struck with a mallet. Handbells struck in this manner are generally not damped," [HMA10].
	mallet on table	"This symbol indicates that the handbell is resting on a padded table and is struck with a mallet," [HMA10].
	mallet lift	"A *mallet lift* is executed by lifting the handbell immediately after striking it with a mallet," [HMA10].
	suspended mallet roll	"A *mallet roll* is similar in technique to a drum roll. Using a mallet in each hand, the ringer rapidly and repeatedly strikes the handbell with mallets while the handbell is held by the handle," [HMA10].
	mallet roll on table	"A *mallet roll* is similar in technique to a drum roll. Using a mallet in each hand, the ringer rapidly and repeatedly strikes the handbell with mallets while the handbell is resting on a padded table," [HMA10].

See section 1.11.2 for proper use of mallets.

Table 22: Notations for the mallet family

pictograph	name	description
▼	martellato	"*Martellato* indicates the handbell is rung by holding it by the handle and gently striking the full body of the handbell horizontally on a properly padded table," [HMA10].
▼	hand martellato	"A *martellato* with the hand on the outside of the handbell casting instead of on the handle," [HMA10].
▼↑	martellato lift	"A *martellato lift* indicates the martellato followed by immediately lifting the handbell to allow the sound to continue," [HMA10].
▼	muted martellato	"A *muted martellato* is a technique used primarily in solo ringing. It is executed by placing one or two fingers on the casting of the handbell while gently striking it on the table," [HMA10].
✱	wall martellato	A *wall martellato* is a technique that should not be used although its effect is second to none. It is executed by crashing a bell on a wall (or floor) with enough force to damage the bell with a unique sound, attracting a significant amount of attention.

See section 1.11.3 for proper use of the martellato technique.

Table 23: Notations for the martellato family

pictograph	name	description
8va	all'ottava	These treble notes are octavated: Instead to the marked notes written in staff, the higher handbells with interval octave are to play. The notation may vary.
8vb	all'ottava bassa	These bass notes are octavated: Instead to the marked notes written in staff, the lower handbells with interval octave are to play. The notation may vary.
Coll'8	coll'ottava	These bass notes are doubled: In addition to the marked notes written in staff, the lower handbells with interval octave are to play. The notation may vary.
Coll'8	coll'ottava	These treble notes are doubled: In addition to the marked notes written in staff, the higher handbells with interval octave are to play. The notation may vary.

The octavation notation is likely used on treble notes higher than B_5 or bass notes lower than C_4, respectively. The all'ottava notation is often used as a synonym for coll'ottava for handbells.

Table 24: Notations for octavation

pictograph	name	description
opt.	optional	"*Optional notes* are written as stemless solid notes in the same size as the required notes. They are designated by a longitude bracket and the word optional or *opt.*," [HMA10]. Likely the sheet contains information about the bell sets, compare section 3.2.
( ) [ ] < >	enclosures for optional notes	These braces are often used throughout all staffs to indicate optional notes. The usage may vary from composition to composition and is specified in the *handbell used chart*, see section 3.2.

Table 25: Notations for optional notes

pictograph	name	description
Pl	pluck	"A *pluck* indicates the *plucking* technique. Handbells are placed on a padded table and sounded by moving the clapper manually," [HMA10].
Pl·↑	pluck lift	"To execute a *pluck lift*, pluck the handbell in normal fashion and immediately lift it from the padded table so it continues to sound," [HMA10].
TPl	tap pluck	"A *tap pluck* indicates that the handbells are placed on a padded table and sounded by tapping the clapper downward with the thumb," [HMA10].

Table 26: Notations for the pluck family

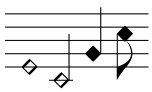
pictograph	name	description
HB	handbells	Play with *handbells*.
HC	handchimes	Play with *handchimes*.
	handchimes	Play with *handchimes*. "Note head shape used for a handchime part to distinguish it from a handbell part when both are notated on the same staff," [HMA10].
R	ring	"A *ring* indicates the normal manner of ringing and damping according to note values. Also, the use of *ring* indicates a return to the normal ringing and damping technique after a passage when other style or technique (such as let vibrate or pluck) has been used. It is understood that handbells are to be rung in normal fashion at the beginning of a piece without the use of the symbol *ring*," [HMA10].

Table 27: Notations for handbells and handchimes

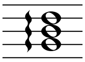
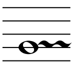
pictograph	**name**	**description**
	rolled chord	"A *rolled chord* may be produced by ringing the notes of the chord in rapid succession from low to high rather than sounding the notes simultaneously," [HMA10]. Sometimes the notation is varied by adding an arrow to the top or the bottom to the arpeggio-line to indicate a rising or descending order.
tr	trill	"A *trill* indicates alternating the ringing of two handbells of adjacent pitches. The simultaneous shaking of two handbells of adjacent pitches is often used to simulate a trill. In handbell notation both pitches may be notated," [HMA10].
Sk *or* 	shake	"The *shake* symbol indicates the rapid shaking of a handbell with the clapper striking both sides of the handbell," [HMA10]. The length of the shake line indicates the length of shaking coinciding with the corresponding note value. The line does not indicate the frequency. When two shake notes in two succeeding measures are tied, the shake techniques is not interrupted. See also *mallet roll* techniques.

Table 28: Notations for the shake family

pictograph	name	description
TD	thumb damp	"A *thumb damp* indicates that the thumb of the hand holding the handbell is placed on the outside of the handbell casting, producing a stopped sound when the clapper strikes the handbell. The addition of one or two fingers on the casting for all but the smallest handbells may be necessary to achieve a completely stopped sound," [HMA10].
HD	hand damp	"A *hand damp* is used with large handbells. When one or two fingers on the casting are insufficient to produce the desired staccato effect, either hand may be placed on the outside of the handbell casting as the handbell is rung. This technique is also useful when a staccato note quickly follows a rung or vice versa," [HMA10].
RT	ring touch	"A *ring touch* indicates that a handbell is rung close to the shoulder with an immediate touch of the handbell to the shoulder to stop the sound," [HMA10].

The techniques thumb damp and hand damp are closely related, each technique is applied to a certain bell range.

Table 29: Notations for stopped rings

pictograph	name	description
CD	controlled diminuendo	"The *controlled diminuendo* is executed after ringing a handbell, particularly a large handbell, slide a gloved hand or finger(s) up the outside of the handbell toward the rim of the casting. Varying degrees of pressure will control the diminuendo. An explanatory footnote may be used," [HMA10].
BD	brush damp	"A *brush damp* means that a ringing handbell is brushed downward against the chest resulting in a sudden reduction of volume. This technique may be used when a soft ring follows a loud ring with the same handbell(s). The effect simulates a forte-piano," [HMA10].
RB	rim brush	"A *rim brush* is performed by ringing notes on beat one, positioning horizontal over the table on beat two, then dragging toward the ringer across the table on beat three," [HMA10]. On beat two and afterwards, the handbell's rim is resting to the table. Some pressure towards the padded table may be necessary.
RD	rim damp	"After ringing the bell on beat 1, one side of the rim of the bell is damped on the table [or by hand — M. Jedamzik] on beat 2, leaving only the harmonics sounding", [Gra95, page 1]. This technique is also used in the piece *Lux Aeterna* by Michael J. Glasgow.

Table 30: Notations for the volume reducing techniques

pictograph	name	description
	bell tree	A *bell tree* consists of “three or more handbells with interlocked handles,” [HMA10]. The bell tree is usually held in one hand or hung from a stand and the bells are usually played with mallets. The bells may be hand-damped or played *let vibrate*.
l	left	“Left hand,” [HMA10].
r	right	“Light hand,” [HMA10].
l-r	pass bell	“Handbell passed from the left hand to the right hand,” [HMA10].
r-l	pass bell	“Handbell passed from the right hand to the left hand,” [HMA10].
$\cancel{\mathsf{B}^\flat_5}$	remove	“Remove $\mathsf{B}^\flat_5$ from table,” [HMA10].
$\mathsf{C}^\sharp_6$ $\cancel{\mathsf{B}^\flat_5}$	replace	“Remove $\mathsf{B}^\flat_5$ from table and put $\mathsf{C}^\sharp_6$ in its place” [HMA10].
E_6 $\wedge$ $\mathsf{E}_5\ \mathsf{F}_5$	place	“Place E_6 in the space above and between E_5 and F_5,” [HMA10].
E_6 $\wedge$ E_5	place	“Place E_6 directly above E_5,” [HMA10].

Table 31: Notations for handbell layout

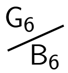
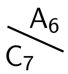
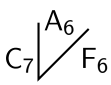
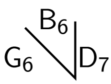
pictograph	**name**	**description**
	left hand: four-in-hand setup	"G_6 is in the primary position and B_6 is in the secondary position: bells ring independently," [HMA10].
	right hand: four-in-hand setup	"A_6 is in the primary position and C_7 is in the secondary position: bells ring independently," [HMA10].
	left hand: shelley setup	"E_6 is in the primary position and E_7 is in the secondary position: bells ring simultaneously," [HMA10].
	right hand: shelley setup	"G_6 is in the primary position and G_7 is in the secondary position: bells ring simultaneously," [HMA10].
	left hand: six-in-hand setup	"F_6 in primary position," [HMA10]. Bells ring independently.
	right hand: six-in-hand setup	"G_6 in primary position," [HMA10]. Bells ring independently.

Table 32: Notations for multiple handbells in hand

Pagina vacat.

Cetervm censeo campanam C_{II} possidendam esse.

from exercitio cottidanio
MICHAELVS CATO

A. One more thing...

This book is donationware or careware: If you like this book or if you want to support the author, please help the last paragraph of the following fairy tale to become true and feel free to make a donation (mentioning something like “donation: Handbell Compendium”) to the *Handglockenchor Wiedensahl* (German, Handbellchoir Wiedensahl). The ensemble has ambitions to upgrade the bell range with the lower seventh octave of Schulmerich bronze handbells bell by bell. These seven handbells are also known as *the magnificent seven.* You can find contact information to do so on their homepage

http://www.handglockenchor-wiedensahl.de/

or [Eic10] and some more information on the ensemble in section 1.4.8.

The Tale of the Seven Bells of the Lower Seventh Octave

Once upon a time a German handbell enſemble from a ſmall, ſmall village traveled weſt to a country far, far away. That very country is known as the **country of unlimited opportunities**. One of the ringers, **Michæl**, a tall boy of young age, ſomehow managed to ring a C_{I} made of the thirteenth element after a concert of a great enſemble. Although impreſſed by the huge ſize of that very inſtrument the young boy was not ſatisfied with the ſound. He thought that thiſ material was excellent for making briefcaſes but not excellent for theſe inſtruments. The young boy noticed that the diſciple of **Schulmerich** alſo manufacture the ſeven baſs bells of the lower ſeventh octave in ſeven huge caſes. His deſire for theſe inſtruments began to grow. Theſe inſtruments are made of the beſt material known and are ſtill ringable. He ſoon was known for his deſire of the bells of huge weight within other members of the enſemble.

After ſeveral years of growing and ringing the young boy became an adult and a paſſionate baſs bell ringer within the enſemble. More than ſeven years later after their firſt trip the handbell enſemble viſited its greateſt friend, a miniſter named **Milton** III. again. The noble **Milton** III. kindly allowed the German enſemble to integrate his ſeven bells into their concert ſuch that **Michæl** was the only one of the German ringers to ring theſe gorgeous bells in their laſt concert of their ſecond concert tour in the country of unlimited opportunities. The outſtanding ſound of theſe heavy baſs bells made a huge contribution to the **March of the Kings** ſuch that the ringers and the director are convinced that theſe ſeven bells are a true contribution. After the enſemble left their friends to finally return home, diſſatisfaction began to grow:

> Erzähle der Welt und dem Himmel die Not:
> Das C_{II} iſt fort.

Some time later ſome gentlemen as well as ſome ladies felt the paſſion of the enſemble and became patronages. Moſt of them donated farthings, groats or gold ſovereigns and were much appreciated by the enſemble. Some even donated a whole bell. Finally the enſemble could upgrade its bell ſet bell by bell to the whole lower ſeventh octave. And there was much rejoicing among the enſemble. All ringers and all patronages lived happily ever after.

KEEP CALM

AND

RING ON

I would like to thank (in alphabetical order) Maximilian Buhr, Pastor Joachim Diestelkamp, Diemut Sigrun Eickhoff (née Lange), Thomas Eickhoff, Nancy Kirkner, Heinrich-Arend Krömer d. Ä., Heinrich-Arend Krömer d. J., Ilse Krömer, Kevin McChesney, Richard Pinkerton, Megan Reishus, C. Milton Rodgers III., Michèle Diana Sharik, Sven Sölter, Marilyn Uzzle, and Jan-Hendrik Zissing for helping me by pointing out mistakes, which allowed me to make corrections. All of them are not responsible for any mistakes still to be found in this book. I would further like to thank every author or institution that granted permission for cited content.

Further thanks are addressed to the *Musikstiftung im Kirchenkreis Stolzenau-Loccum* for collecting donations to fund a Schulmerich bronze F_2 handbell on March 7, 2016.

Für das nächste Stück benötigen wir unseren stärksten Glöckner, um für Sie die kleinsten Glocken zum Klingen zu bringen.

from an early 2000s performance
Thomas Eickhoff

B. Comments and thoughts

B.1. Comments on Professionalism

Since the question on professionalism is more important for a director than for a ringer, because the director formulates the aims of an ensemble including guidelines for ringers, this chapter is part of the appendix. The *three basics of handbell performance* as in appendix B.1.3 are not an exception since these principles are fundamental to (serious) music.

You may do not agree on these thoughts, but I kindly like to ask you to tolerate my view as long as you do not find any major logical errors.

The very inspiring article [Kir11, *Handbell Articles (Blog). Professionalism from May 1, 2013*, *retrieved* September 19, 2013] is recommended for reading and contemplating.

I see no option but to begin this section with a personal and subject statement:

B.1.1. Thoughts about professional handbell ensembles

The first question is not as simple as it may appear: "What is a professional handbell ensemble?" For me a requirement is that all ringers and especially the director are professional musicians; all musicians have graduated in music and are able to play at least one classical orchestra instrument or are professional singers, and all have experiences with interpreting music and conducting handbell ensembles. Every member has additional interests like acoustics, bellfounding or history. The ensemble has an additional non-performing staff like any other professional orchestra with managers, stage workers, public relations managers and others.

This criterion is likely not satisfied by any existing handbell ensemble, although many ensembles and soloists have achieved a magnificent level of skill which is far superior to my personal skills when it comes to handbells. Being an amateur-musician is not a bad thing, especially in the handbell world.

The number of ringers and the handbell assignment are only determined by the simple question "What is best for music?" For sure a small chamber orchestra dedicated to the music of Johann Sebastian Bach or Jan Dismas Zelenka[77] is not unprofessional just because a large symphony orchestra focusing on the music of Gustav Mahler has more than 20 times more musicians. Handbell soloists have to play a different kind of literature than the usual literature written for choirs, and movements like "eight handbell duets" are different interesting attempts which lead to new compositional aspects. I personally think that a large handbell ensemble with a large set of handbells and handchimes and additional sets of *highlighting bells* like Petit & Fritsen or Silver Melody Bells is the best, but not a necessary conceptual base for an outstanding musical experience for the audience. Obviously this does not make one or two performances with only a few ringers like solo, duet or quartet on a concert superfluous. In fact a few small ensembles of highly skilled handbell musicians and a remarkable versatile and entertaining repertoire to give a full-length concert exist.

B.1.2. Personal statement on repertoire

Numerous compositions were written exclusively for handbells. These compositions are usually short in length which is in general not unprofessional.

[77] If his name does not ring a bell, M. Jedamzik urges you to familiarize yourself with his works.

I would like to see a handbell ensemble to play longer compositions for handbells like sonatas, suites or concertos for handbells and chamber orchestra. The challenges encountered with these literature include stamina of the ringers, taking time for rearranging the bell setup and the access to additional skilled musicians and their instruments.

While historically informed performances does not stop on the question of the right instruments, for example harpsichord versus pianoforte, some musicians refuse arrangements in general. The handbell community cannot claim to perform classic pieces authentically in the manner of historically informed performances, but may strive for technical challenging and pleasing arrangements and for good ringing.

B.1.3. What experts say

The area XII of the Handbell Musicians of America asked some prominent handbell experts the questions "What future do you see for handbells? More to the point, what do you think must happen in order to elevate handbell ringing to the level of public support and recognition enjoyed by community orchestras, choral ensembles, and the like?", see [Han13, *History. Voices From the Past.*, *retrieved* August 27, 2013].

Donald E. Allured

Donald E. Allured is cited on [Han13, *History. Voices From the Past. Donald E. Allured*, *retrieved* August 27, 2013] by

> "Handbells are here to stay and will do so well into the future. But acceptability as a legitimate ensemble is still severely limited by lack of attention to the *three basics of handbell performance*: eliminating mistakes, eliminating colorless playing with little or no dynamic change, and ignoring note values with regard to stopping the sound at the right time according to the printed score, in other words, damping at the right time on shoulder or table. Also, Directors and their Choirs need to be more conscious of the visual aspect of their performing, being aware that this is Choreography to good handbell playing. Few musical ensembles really lend themselves to this but handbells do

and choirs need to make much more of it than they normally do. This is where good taste is so important, because movement and its gracefulness purposes to enhance performance, not distract from it. The term 'musical art' is part of our motto and in those works is a built-in standard to be observed by us all."

Richard Coulter

RICHARD COULTER is cited on [Han13, *History. Voices From the Past. Dick Coulter*, *retrieved* August 27, 2013] by

"It is difficult to know what we can do in order to bring handbells into the real arena of 'acceptable' music – similar to choruses and orchestras which are supported by local communities. We need to get away from playing music which is either:

1) too showy with little content *or*

2) arranged from some other medium.

Sonos has tried to lead the way and is, apparently, succeeding to some extent. I think we need to encourage the composition of a whole lot more music for handbells that is 'unique' but not necessarily 'avant-garde', that is easily understood without being trite or simple, and something which sings with beautiful phrasing, exciting harmonies and which will challenge the listeners' ears and minds.

As I have pondered this last question a little more, it seems to me that we might have to get to the point of having many more ringers in an organization than we do now – similar to the manner in which some Japanese conductors have developed their groups. After all, an orchestra has several violinists – why shouldn't a handbell group have many more ringers. By doing that, we could perhaps develop the ability to perform deeper and larger works. I would have to think more about that."

B.1.4. One note on aluminium handbells

The main instruments of a handbell ensemble are handbells as in section 1.3. Some likewise advanced handbell ensembles use handchimes, Dutch handbells or Silver Melody Bells occasionally highlight a certain voice or musical lines due to a noticeable audible difference from handbells. Another potential is the use of *tubular bells* as used in orchestras.

Aluminium bass handbells were developed as an alternative to heavy bronze bass handbells. The weight of bronze bass handbells makes them practically impossible to ring for some ringers, especially ringing multiple bells in a row. Aluminium bells have a strong fundamental tone, see [Mal13a, *Tradition. Innovations*, *retrieved* August 5, 2013]; Their sound differs from bronze handbells. This does not affect their use for bass lines. Another aspect is mixing bronze and aluminium bells for the same bass line or voice.[78]

An example is Johann Sebastian Bach's passacaglia. An excerpt with an octavated bass line is given by figure 22. The upper bass line is typically rung with bronze bells. Basically a good idea is to ring the lower bass line with aluminium bells. The note C_3 is used in anacrusis in the lower bass line as well as in bar 8 in the upper bass line. This implies that the ensemble would need two C_3 handbells: one made of bronze and one made of aluminium. This also holds for F_3 and G_3. Table 2a shows that not all aluminium handbells from the lower bass line are available but all are available in bronze regularly. This is likely a special characteristic of this particular piece.

There are numerous examples of ensembles using Basso Profundo bells along with bronze bells in the same bass lines. Not everyone dislikes this practice of inconsistency related to musical lines. On the other hand, this consistency obviously causes additional costs. Mixing handbells in musical lines is a better solution than omitting them. But this does not mean that this solution is optimal.

[78] Οὖτις excludes some notes of a clarinet and assigns these notes to a saxophone.

B.1.5. Contact with the audience

My personal view is without a doubt molded by my personal experience in Germany while taking my humble part in the Handglockenchor Wiedensahl. A part of the success of our ensemble surely is the fact that handbell ensembles and our instruments are unknown to most people, thus making handbell music even more attractive. Our director Thomas Eickhoff guides the audience through the concerts with information about the history of handbell music, the instruments itself, history of our ensemble, pieces to perform and related topics. He also invites the audience to see the instruments after the concert. Many audience members are very curious about our instruments and talk to the ringers and want to try ringing the bells after a short instruction by ringers. Some of them ask really good questions. These after show talks are enjoyed by some ringers, too. It is a nice experience to see the excitement and astonishment of the audience. This is an aspect of Nancy Kirkner's statement that I disagree[79] with (at least from someone, who lives in a country, where the existence of handbell music is mostly unknown):

> "In general, handbell musicians haven't embraced the same disciplines that shape professional musicians in other areas, and haven't put in the work to earn a large following. [… — M. Jedamzik].
>
> Seldom will a professional classical musician in concert:
>
> - [… — M. Jedamzik].
> - Explain techniques and take questions from the audience.
> - Invite the audience to come on stage and try out their instrument.
>
> We see these things in handbells all the time, even among elite ringers, and (in my opinion) they detract from the image we want to cultivate. If we want to be taken seriously, we need to play by the same rules as professional musicians.

[79] In my opinion, the other items of her list are indeed questionable for an advanced handbell ensemble.

> It's fine to play handbells for our own enjoyment, but let's not expect people to give up an afternoon or evening to watch us have a good time. People attend concerts so they can have a good time, and that results from the quality of the musicians and the experience offered. Let's not waste time solving the wrong problem.", [Kir11, *Handbell Articles (Blog). Professionalism from May 1, 2013*, *retrieved* September 19, 2013].

While the cited two points of the list are true, the special situation of an unknown art form in countries like Germany requires a different treatment. A usual audience member does not harm either a handbell or a handchime when instructed correctly by a ringer. This may also differ from usual orchestra instruments. These actions and questions take place after the concert to ensure an uninterrupted concert flow. My personal impression is that a friendly contact with the audience also makes handbell ensembles appear more likable and thus makes the concert for the audience a better experience. I do not see that the written behavior is obstructive for an amateur ensemble which wants to be taken seriously.

B.2. Issues and research

As stated in the preface, this book covers many topics of my personal interest. However, there are aspect of my interest which are not the covered by this survey. The following list covers some of them. To stress particular aspects, some questions are added. Some of these questions might already be partially answered, but the results might not be freely accessible.

(a) History

  (i) How did the statistic aspects

  - number of handbell ensembles with number of octaves,
  - number of handbells sold,
  - number of schools using handbells or handchimes,
  - number of published sheet music and sales

  develop over time?

(ii) How did the handbell music itself develop over time?

(iii) What kind of influence do symposia and conventions have on the handbell world?

(iv) Apply scientific research on the history of handbells and compile an in-depth study.

(b) Physics

(i) Refine the models discussed in this book.

(ii) How does ringing a bell determine the boundary conditions of the wave equation?

(iii) Investigate the questions in remark 66 and apply recent scientific results.

(iv) Is there a way to quantify the change of a bell's tone by tarnishing and polishing? How do theses effect the whole bell set?

(v) Apply numerical analysis on the handbell wave equation.

(vi) What kind of architectural acoustics are good for handbells in concert?

(vii) Investigate psychoacoustics related to handbell music.

(c) Composing and Arranging

(i) Are there any good ways to transcript a certain instrument to handbells? If so, can a software which covers mechanical part of this labor be developed?

(ii) Is there a metric to decide, whether a piece of music can be performed on handbells appealingly?

(iii) Is there a way to classify original handbell compositions? A classification by level or composer is trivial.

(iv) Are there any new techniques?

(d) Management and marketing

(i) What needs to be done to make handbells more popular in areas, where handbells are almost unknown, and establish new handbell ensembles in these areas?

   (ii) Can a professional handbell ensemble as in appendix B.1.1 in which all of its members and staff personal are payed a reasonable wage and which performs on its own worldwide concert tours be established?

(e) Education and pedagogy

   (i) What is the best way of teaching certain groups of people of certain background (age, musical background) to ring handbells?

   (ii) Are there good ways to apply handbells (or handchimes) in musical education?

   (iii) Does handbell ringing have an influence on the psychological or physical development of the ringer as a human being?

(f) An outline of other interesting topics can be found in section 4.3.

Pagina vacat.

...mehr denn dreihundert, mit Beilen und Pechkränzen versehene Bösewichter, aus den Mauern unserer damals irregeleiteten Stadt, erwarteten nichts als das Zeichen, das der Prädikant geben sollte, um den Dom der Erde gleich zu machen. Dagegen, bei Anhebung der Musik, nehmen Eure Söhne plötzlich, in gleichzeitiger Bewegung, und auf eine uns auffallende Weise, die Hüte ab, sie legen, nach und nach, wie in tiefer unaussprechlicher Rührung, die Hände vor ihr herabgebeugtes Gesicht...

from *Die Heilige Cäcile* oder *Die Gewalt der Musik*
Bernd Heinrich Wilhelm von Kleist

C. Handbell poetry

As seen in section 1.4 bells were used as instruments for centuries. The use of bells inspired poets to write poetry. Handbells also inspired ringers to become amateur-poets. A remarkable result is the book [McC11] which is available on https://sonologymusic.com/products/limericks-lessons-and-life-in-handbells/ for ordering.

This chapter is for those who cannot get enough and want an encore.

Please note that the the authors are the corresponding copyright owners. Everyone of them kindly allowed me to print their works.

The webpage [BBA94, *Hymns for Ringers*, *retrieved* March 11, 2017] hosts some hymns.

C.1. Poetry on Ringers

C.1.1. Handbell Notes

by Raymond A. Foss from [Fos06, *retrieved* September 02, 2014]

Watching them play
Seeing their notes on the paper
Sensing the fun in practice
in hearing the sheet come alive
Feeling a shake
Never realizing the joy
Captured in the dots and waves
on the printed page.

C.1.2. A Bell Ringer's Confession

by Anonymous from [OnL01, *A Bellringer's Confession*, *retrieved* June 9, 2001]

Almighty and most merciful Conductor,
We have erred and strayed from Thy beat like lost sheep;
We have followed too much the accidentals and tempi
    of our own hearts.
We have offended against Thy dynamic markings.
We have left unrung those notes which we ought to have rung
And we have rung those notes which we ought not to have rung
And there is no damping in us.

But Thou, O Conductor, have mercy upon us, miserable ringers;
Succor the key-signature challenged;
Restore Thou them that need help turning their page;
Spare Thou them that have pencils.
Pardon our mistakes, and have faith that hereafter
We will follow Thy directions
And ring together in perfect harmony.

C.1.3. People

by Kevin McChesney from [McC11, page 74]

The bells in themselves may be elegant.
You ring amidst laughter and merriment.
The very best part
Of bells is the heart –
The *people* are really the instrument!

C.1.4. This is a belfry that is free

by Anonymous from [All04, *Bells*, *retrieved* March 26, 2014]

This is a belfry that is free
For all those that civil be
And if you please to chime or ring
It is a very pleasant thing

There is no musick played or sung
Like unto bells when they're well rung
Then ring your bells well if you can
Silence is best for everyman

But if you ring in spur or hat
Sixpence you pay, be sure of that
And if a bell you overthrow
Pray pay a groat before you go.

C.1.5. 'Twas the Night Before Rehearsal

by Karen Paschke from [MACH05, page 6]

'Twas the night before rehearsal and all through my flat
Not a creature was stirring, not even my cat.
My papers were graded, put away in my pack
And I had finally hit the sack.

In my flannel pajamas decorated with a cow
I was not thinking of music just now.
When suddenly, in the midst of my slumbers
A vision of Visions through my head lumbers.

I ring five bells and six handchimes
And it changes keys one dozen times! (Yes. I counted.)
It has weaves, it has accidentals, it has sixteenth notes,
It has places where it has gotten my goat.

I can mark the music, I can dog-ear the pages,
I can practice page seven for what seems like ages.
There are two problems to be solved,
    two mountains to overcome,
The first is to make those sixteenth notes run.

Inertia is real, those bells have weight
And sixteenths are twice as fast as eighths.
But they do not overpower me, I work out now and then.
The problem I must overcome is "When."

When the sixteenth notes up and down the scale run
The thirteen of us must sound like one.
And so my notes do not stick out
I may need to find a new way to count.

The problem's not solved, but I have something to try
In order to make those bells really fly.
I continue to sleep and my dreams shift
To my next problem two pages after the fifth.

My problem here is of a different kind –
I'm constantly changing from bells to chimes.
And added to the problem is the fact
That G-sharp is the same bell as A-flat.[80]

But there is a solution that just might work
Without driving me berserk.
With chimes always in the left hand and bells
    in the right
Maybe everything will come out all right.

The time is now twenty to six
And hopefully my problems I've fixed.
I can stop dreaming of music, for rehearsal I'm ready
And now I can dream about my upcoming trip to the Serengeti.

[80] Karen Paschke changed this line from "That G-sharp is the same as A-flat." in August, 2016. She also considered "note" instead of "bell".

C.2. Poetry on Techniques

C.2.1. Bell hogs (first third)

by Kevin McChesney from [McC11, page 41]

An experienced ringer can bring
Many assets to music we ring,
But we don't keep score
And see who rings more.
To be a "bell hog's" not the thing!

C.2.2. Attend To Technique

by Kevin McChesney from [McC11, page 9]

For confidence and skill to peak
And best artistry, don't you sneak
Past the basics you need
For music to succeed.
Be sure that you know your technique.

C.3. Religion-based poetry

C.3.1. Bell Doxology

by Suzy Gazlay *and* Bill Ingram from [OnL01, *Bell Doxology*, *retrieved* March 3, 2001]

Praise God for those who ring these bells!
Their faithfulness commitment tells:
With joyful effort learn their parts;
They love and serve with all their hearts.

Praise God for those who ring these bells!
A love of music in them dwells;
With every note (and miss!) and chord,
An offering unto the Lord!

Praise God for those who gave these bells,
In whom the gift of giving dwells;
May they be used for many days
To ring our great Creator's praise.[81]

C.3.2. Ye Ringers All

by Anonymous from [Rin09, *Why Ring in Church?*, *retrieved* March 26, 2014]

Ye Ringers all, that do come here,
Give head and hand and heart:
The head for will, the hand for skill,
The heart for worship's part.

[81] Bill Ingram added this last stanza.

C.3.3. Psalm 150

The left column is a transcription of the psalm 150 as found in the King James Bible [Bib11] of 1611 and the right column is a transcription of the same psalm as as found in Dr. Martin Luther's translation [Lut45] of 1545. Typefaces, line breaks and hyphenation do not coincide with the original, hence it is not a facsimile, although the differentiation of blackletter typefaces and serif typefaces coincides. Both translations are often praised for their writing style and accounted for their cultural impact.

PSAL. CL.

An exhortation to praiſe God, 3 with all kind of inſtruments.

[a]PRaiſe ye the LORD. Praiſe God in his Sanctuarie: Praiſe him in the firmament of his power.
2 Praiſe him for his mightie actes: Praiſe him according to his excellent greatneſſe.
3 Praiſe him with the ſound of the [b]Trumpet: Prayſe him with the Pſalterie and Harpe.
4 Praiſe him with the timbrell and [c]dance: praiſe him with ſtringed inſtruments, and Organes.
5 Praiſe him vpon the loud cymbals: praiſe him vpon the high ſounding cymbals.
6 Let euery thing that hath breath, praiſe the LORD. Praiſe yee the LORD.

[a] *Halleluiah.*
[b] *Or, Cornet.*
[c] *Or, Pipe.*

CL.

Halelu ia.

LObet den HERRN in ſeinem Heiligthum / Lobet jn in der Feſte ſeiner Macht.
Lobet jn in ſeinen Thaten / Lobet jn in ſeiner groſſen Herrligkeit.
Lobet jn mit Poſaunen / Lobet jn mit Pſalter vnd Harffen.
Lobet jn mit Paucken vnd Reigen / Lobet jn mit Seiten vnd Pfeiffen.
Lobet jn mit hellen Cymbeln / Lobet jn mit wolklingenden Cymbeln.
ALles was Odem hat / Lobe den HERRN / Halelu ia.

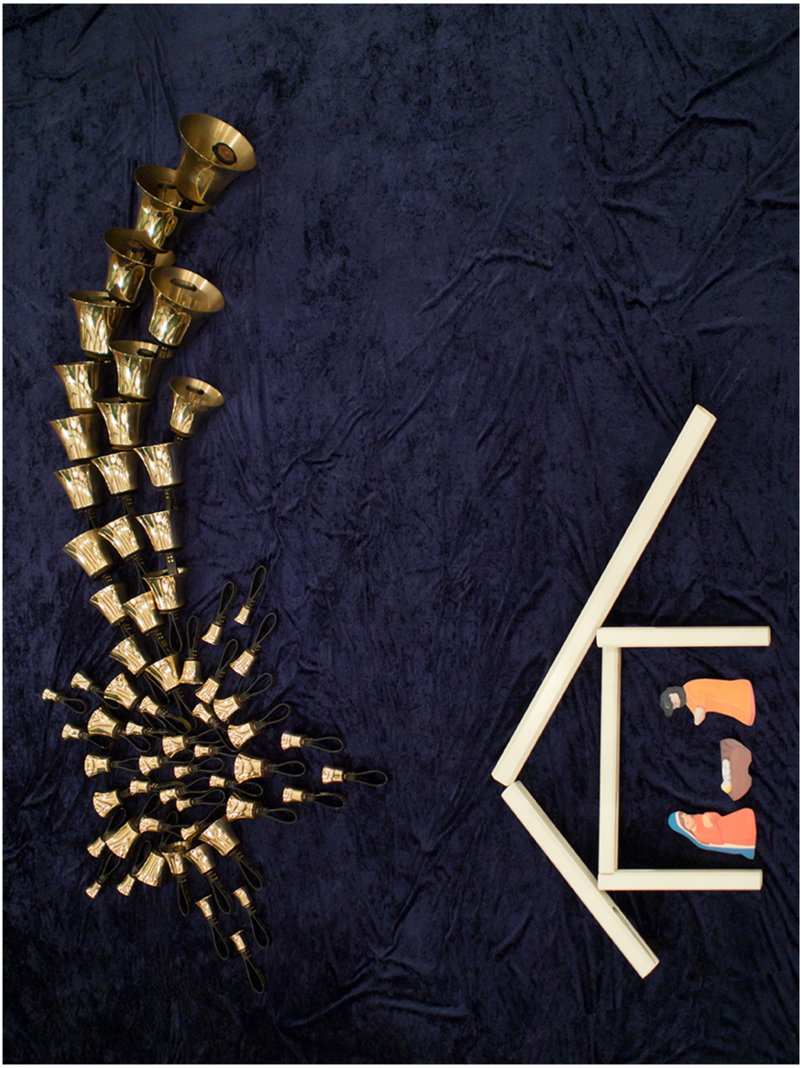
(a) A Christmas bell tree

This picture was uploaded to *facebook* on December 22, 2014 to greet all friends of the *Handglockenchor Wiedensahl* and was viewed by over 5 200 users. Idea by THOMAS EICKHOFF, photography and postproduction by MICHAEL JEDAMZIK.

Figure 31: Christmas Bell Tree

This picture was uploaded to *facebook* on December 23, 2015 to greet all friends of the *Handglockenchor Wiedensahl* and was viewed by over 10 900 users. Idea by Thomas Eickhoff, photography and postproduction by Michael Jedamzik.

Figure 32: Bells' Angel



This picture was uploaded to *facebook* on December 23, 2016 to greet all friends of the *Handglockenchor Wiedensahl* and was viewed by over 7 100 users. Idea by Thomas Eickhoff, photography and postproduction by Michael Jedamzik.

Figure 33: Bellstar Of Bethlehem

Pagina vacat.

In bunten Bildern wenig Klarheit,
Viel Irrtum und ein Fünkchen Wahrheit,
So wird der beste Trank gebraut,
Der alle Welt erquickt und auferbaut.

from Faust I. Vorspiel auf dem Theater
JOHANN WOLFGANG VON GOETHE

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from On transferring Information
GILLIAN DOE

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Was andere hierzu getrieben,
sei kurz notiert und aufgeschrieben.

from Galerkins Lösungsannäherungen bei monotonen Abbildungen
Friedrich Wille

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Ring loudly and carry a big handbell; you will go far.

from letter to Henry L. Shake,
January 26, 1900
Theodore Roosebell

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A Handbell Compendium

a summary by Michael Jedamzik

Foreword by Kevin McChesney

This book is divided into five chapters: about handbells, related physics, compositions, handbell ensembles and notation. The composer and handbell clinician Kevin McChesney wrote in the foreword: “I have been delighted to meet Michael through the global handbell network and through reading his ‘Handbell Compendium’, I have been educated by his extensive and earnest research. He has compiled information from a huge number of sources that will interest scientists, musicians in endeavors other than handbells, handbell leaders, ringers, and listeners. I have been impressed at his creation of a reference work that belongs in the library of everyone who leads or just plain loves handbells.” On http://www.handglockenchor-wiedensahl.de/ you can download this book.

Michael Jedamzik began ringing handbells in 2001 and participates in the *Handglockenchor Wiedensahl* (German, “Handbell Choir Wiedensahl”). In 2012 and 2016 the Handglockenchor Wiedensahl became a laureate of the *Deutscher Orchesterwettbewerb* (German, “German Orchestra Competition”) by *Deutscher Musikrat* (German, “German Music Council”) in a row.